

# A General Maximum Principle for Mean-field Stochastic Differential Equations with Jump Processes

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**Abstract.** *In this paper, we investigate the optimal control problems for stochastic differential equations (SDEs in short) of mean-field type with jump processes. The control variable is allowed to enter into both diffusion and jump terms. This stochastic maximum principle differs from the classical one in the sense that here the first-order adjoint equation turns out to be a linear mean-field backward SDE with jumps, while the second-order adjoint equation remains the same as in Tang and Li's stochastic maximum principle [32]. Finally, for the reader's convenience we give some analysis results used in this paper in the Appendix.*

**Keywords:** *Mean-field SDEs. Jump processes. Optimal stochastic control. Maximum principle. Spike variation method. McKean-Vlasov equations.*

**AMS Subject Classification:** 60H10, 93E20.

## 1. Introduction

In this paper we study stochastic optimal control for a system governed by nonlinear SDEs of mean-field type, which is also called McKean-Valasov equations, with jump processes:

$$\begin{cases} dx^u(t) = f(t, x^u(t), \mathbb{E}(x^u(t)), u(t)) dt + \sigma(t, x^u(t), \mathbb{E}(x^u(t)), u(t)) dW(t) \\ \quad + \int_{\Theta} g(t, x^u(t-), u(t), \theta) N(d\theta, dt), \\ x^u(s) = \zeta, \end{cases} \quad (1.1)$$

where the coefficients  $f$  and  $\sigma$  depend on the state of the solution process as well as of its expected value and the initial time  $s$  and the initial state  $\zeta$  of the system are fixed,

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$(W(t))_{t \in [s, T]}$  is a standard *one*-dimensional Brownian motion and  $N(d\theta, dt)$  is a Poisson martingale measure with characteristic  $\mu(d\theta)dt$ . This mean-field jump diffusion processes are obtained as the mean-square limit, when  $n \rightarrow +\infty$  of a system of interacting particles of the form

$$\begin{aligned} dx_n^{j,u}(t) &= f\left(t, x_n^{j,u}(t), \frac{1}{n} \sum_{i=1}^n x_n^{i,u}(t), u(t)\right) dt \\ &+ \sigma\left(t, x_n^{j,u}(t), \frac{1}{n} \sum_{i=1}^n x_n^{i,u}(t), u(t)\right) dW^j(t) \\ &+ \int_{\Theta} g\left(t, x_n^{j,u}(t-), u(t), \theta\right) N^j(d\theta, dt). \end{aligned}$$

Our control problem consists in minimizing a cost functional of the form:

$$J^{s,\zeta}(u(\cdot)) = \mathbb{E} \left[ h(x^u(T), \mathbb{E}(x^u(T))) + \int_s^T \ell(t, x^u(t), \mathbb{E}(x^u(t)), u(t)) dt \right]. \quad (1.2)$$

This cost functional is also of mean-field type, as the functions  $h$  and  $\ell$  depend on the marginal law of the state process through its expected value.

An admissible control  $u(\cdot)$  is an  $\mathcal{F}_t$ -adapted and square-integrable process with values in a nonempty subset  $\mathbb{A}$  of  $\mathbb{R}$ . We denote the set of all admissible controls by  $\mathcal{U}$ . Any admissible control  $u(\cdot) \in \mathcal{U}$  satisfying

$$J^{s,\zeta}(u^*(\cdot)) = \min_{u(\cdot) \in \mathcal{U}} J^{s,\zeta}(u(\cdot)), \quad (1.3)$$

is called an optimal control. The corresponding state process, solution of SDE-(1.1), is denoted by  $x^*(\cdot) = x^{u^*}(\cdot)$ .

The modern optimal control theory has been well developed since early 1960s, when Pontryagin et al., [24] published their work on the maximum principle and Bellman [6] put forward the dynamic programming method. The pioneering works on the stochastic maximum principle was written by Kushner ([9],[10]). Since then there have been a lot of works on this subject, see for instance ([25],[36],[2],[8],[11],[15],[17]). Peng [25] obtained the optimality stochastic maximum principle for the general case. A good account and an extensive list of references on stochastic optimal control can be founded in Yong et al., [34]. The stochastic optimal control problems for jump processes has been investigated by many authors, see for instance, ([8],[13],[23],[26],[32],[16],[27],[28],[31]). The stochastic maximum principle for jump diffusion in general case, where The control domain need not be convex. and the diffusion coefficient depends explicitly on the control variable, was derived via spike variation method by Tang et al., [32], extending the Peng's stochastic maximum principle of optimality [25]. These conditions are described in terms of two adjoint processes, which are linear classical backward SDEs. The sufficient conditions for optimality was obtained by Framstad et al., [13].

Historically, the SDE of Mean-field type was introduced by Kac [14] in 1956 as a stochastic model for the Vlasov-kinetic equation of plasma and the study of which was initiated by

McKean [21] in 1966. Since then, many authors made contributions on SDEs of mean-field type and applications, see for instance, ([1],[4],[5],[7],[14],[20],[22],[33],[30],[35]). Mean-field stochastic maximum principle of optimality was considered by many authors, see for instance ([5],[20],[22],[35]). In Buckdahn et al., [4] the authors obtained mean-field backward stochastic differential equations. In a recent paper by Buckdahn et al., [5], the maximum principle was introduced for a class of stochastic control problems involving SDEs of mean-field type, where the authors obtained a stochastic maximum principle differs from the classical one in the sense that the first-order adjoint equation turns out to be a linear mean-field backward SDE, while the second-order adjoint equation remains the same as in Peng's stochastic maximum principle [25]. In Mayer-Brandis et al., [22] a stochastic maximum principle of optimality for systems governed by controlled Itô-Levy process of mean-field type is proved by using Malliavin calculus. The local maximum principle of optimality for Mean-field stochastic control problem has been derived by Li [20]. The linear-quadratic optimal control problem for mean-field SDEs has been studied by Yong [35].

Our purpose in this paper is to establish necessary conditions of optimality for Mean-field SDEs with jumps processes, in which the coefficients of diffusion depend on the state of the solution process as well as of its expected value. Moreover, the cost functional is also of Mean-field type. The proof of our main result is based on spike variation method. This results is an extension of *Theorem 2.1* in Buckdahn et al., [5] to the controlled mean-field SDEs with jump processes. To streamline the presentation, we only consider the one dimensional case.

The rest of the paper is organized as follows. Section 2 begins with a general formulation of a mean-field control problem with jump processes and give the notations and assumptions used throughout the paper. In Sections 3 we prove our main result.

## 2. Assumptions and statement of the control problem

Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$  be a fixed filtered probability space equipped with a  $\mathbb{P}$ -completed right continuous filtration on which a  $d$ -dimensional Brownian motion  $W = (W(t))_{t \in [0, T]}$  is defined. Let  $\eta$  be a homogeneous  $(\mathcal{F}_t)$ -Poisson point process independent of  $W$ . We denote by  $\tilde{N}(d\theta, dt)$  the random counting measure induced by  $\eta$ , defined on  $\Theta \times \mathbb{R}_+$ , where  $\Theta$  is a fixed nonempty subset of  $\mathbb{R}$  with its Borel  $\sigma$ -field  $\mathcal{B}(\Theta)$ . Further, let  $\mu(d\theta)$  be the local characteristic measure of  $\eta$ , i.e.  $\mu(d\theta)$  is a  $\sigma$ -finite measure on  $(\Theta, \mathcal{B}(\Theta))$  with  $\mu(\Theta) < +\infty$ . We then define

$$N(d\theta, dt) = \tilde{N}(d\theta, dt) - \mu(d\theta) dt,$$

where  $N$  is Poisson martingale measure on  $\mathcal{B}(\Theta) \times \mathcal{B}(\mathbb{R}_+)$  with local characteristics  $\mu(d\theta) dt$ . We assume that  $(\mathcal{F}_t)_{t \in [0, T]}$  is  $\mathbb{P}$ -augmentation of the natural filtration  $(\mathcal{F}_t^{(W, N)})_{t \in [s, T]}$  defined as follows

$$\mathcal{F}_t^{(W, N)} = \sigma(W(r) : s \leq r \leq t) \vee \sigma\left(\int_s^r \int_{\mathcal{B}} N(d\theta, d\tau) : s \leq \tau \leq t, B \in \mathcal{B}(\Theta)\right) \vee \mathcal{G},$$

where  $\mathcal{G}$  denotes the totality of  $\mathbb{P}$ -null sets, and  $\sigma_1 \vee \sigma_2$  denotes the  $\sigma$ -field generated by  $\sigma_1 \cup \sigma_2$ .

**Basic notations.** For convenience, we will use the following notations throughout the paper. Let  $u(\cdot) \in \mathcal{U}$  be an admissible control. For  $\Phi = f, \sigma, \ell$  :

1.  $\delta\Phi(t) = \Phi(t, x^*(t), \mathbb{E}(x^*(t)), u(t)) - \Phi(t, x^*(t), \mathbb{E}(x^*(t)), u^*(t)).$
2.  $\Phi_x(t) = \frac{\partial\Phi}{\partial x}(t, x^*(t), \mathbb{E}(x^*(t)), u^*(t)), \Phi_y(t) = \frac{\partial\Phi}{\partial y}(t, x^*(t), \mathbb{E}(x^*(t)), u^*(t)).$
3.  $g_x(t, \theta) = g_x(t, x(t_-), u(t), \theta), g_{xx}(t, \theta) = g_{xx}(t, x(t_-), u(t), \theta).$
4.  $\Phi_{xx}(t) = \frac{\partial^2\Phi}{\partial x^2}(t, x^*(t), \mathbb{E}(x^*(t)), u^*(t)), \Phi_{yy}(t) = \frac{\partial^2\Phi}{\partial y^2}(t, x^*(t), \mathbb{E}(x^*(t)), u^*(t)).$
5.  $\Phi_{xy}(t) = \frac{\partial^2\Phi}{\partial x\partial y}(t, x^*(t), \mathbb{E}(x^*(t)), u^*(t)).$
6.  $\mathcal{L}_t(\Phi, y) = \frac{1}{2}\Phi_{xx}(t, x^*(t), \mathbb{E}(x^*(t)), u^*(t))y^2, \quad \mathcal{L}_{t,\theta}(g, y) = \frac{1}{2}g_{xx}(t, x^*(t), u^*(t), \theta)y^2.$
7. We denote by  $\mathbf{I}_A$  the indicator function of  $A$  and by  $\text{sgn}(\cdot)$  the sign function.
8. We denote by  $\mathbb{L}_{\mathcal{F}}^2([s, T]; \mathbb{R}) = \{\phi(\cdot) := \phi(t, w) \text{ is an } \mathcal{F}_t - \text{adapted } \mathbb{R} - \text{valued measurable process on } [s, T] \text{ such that } \mathbb{E}\left(\int_s^T |\phi(t)|^2 dt\right) < \infty\}, \text{ and by } \mathbb{M}_{\mathcal{F}}^2([s, T]; \mathbb{R}) = \{\phi(\cdot) := \phi(t, \theta, w) \text{ is an } \mathcal{F}_t - \text{adapted } \mathbb{R} - \text{valued measurable process on } [s, T] \times \Theta \text{ such that } \mathbb{E}\left(\int_s^T \int_{\Theta} |\phi(t, \theta)|^2 \mu(d\theta) dt\right) < \infty\}.$
9. In what follows,  $C$  and  $\rho(\varepsilon)$  represents a generic constants, which can be different from line to line.

**Basic assumptions.** Throughout this paper we assume the following.

- (H1)** The functions  $f(t, x, y, u) : [s, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{A} \rightarrow \mathbb{R}$ ,  $\sigma(t, x, y, u) : [s, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{A} \rightarrow \mathbb{R}$ ,  $\ell(t, x, y, u) : [s, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{A} \rightarrow \mathbb{R}$  and  $h(x, y) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  are twice continuously differentiable with respect to  $(x, y)$ . Moreover,  $f, \sigma, h$  and  $\ell$  and all their derivatives up to second-order with respect to  $(x, y)$  are continuous in  $(x, y, u)$  and bounded.
- (H2)** The function  $g : [s, T] \times \mathbb{R} \times \mathbb{A} \times \Theta \rightarrow \mathbb{R}$  is twice continuously differentiable in  $x$ , Moreover  $g_x$  is continuous,  $\sup_{\theta \in \Theta} |g_x(t, \theta)| < +\infty$  and there exists a constant  $C > 0$  such that

$$\sup_{\theta \in \Theta} |g(t, x, u, \theta) - g(t, x', u, \theta)| + \sup_{\theta \in \Theta} |g_x(t, x, u, \theta) - g_x(t, x', u, \theta)| \leq C|x - x'| \quad (2.1)$$

$$\sup_{\theta \in \Theta} |g(t, x, u, \theta)| \leq C(1 + |x|). \quad (2.2)$$

Under the above assumptions, the SDE-(1.1) has a unique strong solution  $x^u(t)$  which is given by

$$\begin{aligned} x^u(t) = & \zeta + \int_s^t f(r, x^u(r), \mathbb{E}(x^u(r)), u(r)) dr + \int_s^t \sigma(r, x^u(r), \mathbb{E}(x^u(r)), u(r)) dW(r) \\ & + \int_s^t \int_{\Theta} g(t, x^u(r_-), u(r), \theta) N(d\theta, dr), \end{aligned}$$

and by standard arguments it is easy to show that for any  $q > 0$ , it holds that

$$\mathbb{E}(\sup_{t \in [s, T]} |x^u(t)|^q) < C_q, \quad (2.3)$$

where  $C_q$  is a constant depending only on  $q$  and the functional  $J^{s, \zeta}$  is well defined.

**Usual Hamiltonian.** We define the usual Hamiltonian associated with the mean-field stochastic control problem (1.1)-(1.2) as follows

$$\begin{aligned} H(t, X, \mathbb{E}(X), u, \Psi(t), K(t), \gamma_t(\theta)) = & \Psi(t) f(t, X, \mathbb{E}(X), u) \\ & + K(t) \sigma(t, X, \mathbb{E}(X), u) + \int_{\Theta} \gamma_t(\theta) g(t, x(t), u(t), \theta) \mu(d\theta) \\ & - \ell(t, X, \mathbb{E}(X), u), \end{aligned} \quad (2.4)$$

where  $(t, X, u) \in [s, T] \times \mathbb{R} \times \mathbb{A}$ ,  $X$  is a random variable such that  $X \in \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{R})$  and  $(\Psi(t), K(t), \gamma_t(\theta)) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$  given by equation (2.5).

**Adjoint equations for mean-field SDEs with jump processes.** We introduce the adjoint equations involved in the stochastic maximum principle for our control problem. The first-order adjoint equation turns out to be a linear mean-field backward SDE with jump terms, while the second-order adjoint equation remains the same as in Tang et al., [32].

For any  $u(\cdot) \in \mathcal{U}$  and the corresponding state trajectory  $x(\cdot)$ , we define the first-order adjoint process  $(\Psi(\cdot), K(\cdot), \gamma(\cdot))$  and the second-order adjoint process  $(Q(\cdot), R(\cdot), \Gamma(\cdot))$  as the ones satisfying the following equations:

1. *First-order adjoint equation: linear backward SDE of mean-field type with jump processes*

$$\left\{ \begin{aligned} d\Psi(t) = & - \left\{ f_x(t) \Psi(t) + \mathbb{E}(f_y^\top(t) \Psi(t)) + \sigma_x(t) K(t) \right. \\ & + \mathbb{E}(\sigma_y(t) K(t)) + \ell_x(t) + \mathbb{E}(\ell_y(t)) \\ & \left. + \int_{\Theta} g_x(t, \theta) \gamma_t(\theta) \mu(d\theta) \right\} dt \\ & + K(t) dW(t) + \int_{\Theta} \gamma_t(\theta) N(dt, d\theta) \\ \Psi(T) = & - (h_x(x(T), \mathbb{E}(x(T))) + \mathbb{E}(h_y(x(T), \mathbb{E}(x(T))))). \end{aligned} \right. \quad (2.5)$$

**2. Second-order adjoint equation: classical linear backward SDE with jump processes**  
(see Tang et al., [32] equation (2.23))

$$\left\{ \begin{array}{l} dQ(t) = - \left\{ 2f_x(t) Q(t) + \sigma_x^2(t) Q(t) + 2\sigma_x(t) R(t) \right. \\ \quad \left. + \int_{\Theta} (\Gamma_t(\theta) + Q(t)) (g_x(t, \theta))^2 \mu(d\theta) + 2 \int_{\Theta} \Gamma_t(\theta) g_x(t, \theta) \mu(d\theta) \right. \\ \quad \left. + H_{xx}(t) \right\} dt + R(t) dW(t) + \int_{\Theta} \Gamma_t(\theta) N(d\theta, dt) \\ Q(T) = -h_{xx}(x(T), \mathbb{E}(x(T))) \end{array} \right. \quad (2.6)$$

**Remark 2.1.** As it is well known that under conditions (H1) and (H2) the first-order adjoint equation (2.5) admits one and only one  $\mathcal{F}_t$ -adapted solution pair  $(\Psi(\cdot), K(\cdot), \gamma(\cdot)) \in \mathbb{L}_{\mathcal{F}}^2([s, T]; \mathbb{R}) \times \mathbb{L}_{\mathcal{F}}^2([s, T]; \mathbb{R}) \times \mathbb{M}_{\mathcal{F}}^2([s, T]; \mathbb{R})$ . This equation reduces to the standard one as in (Tang et al., [32] equation (2.22)), when the coefficients not explicitly depend on the expected value (or the marginal law) of the underlying diffusion process. Also the second-order adjoint equation (2.6) admits one and only one  $\mathcal{F}_t$ -adapted solution pair  $(Q(\cdot), R(\cdot), \Gamma(\cdot, \theta)) \in \mathbb{L}_{\mathcal{F}}^2([s, T]; \mathbb{R}) \times \mathbb{L}_{\mathcal{F}}^2([s, T]; \mathbb{R}) \times \mathbb{M}_{\mathcal{F}}^2([s, T]; \mathbb{R})$ . Moreover when the jump coefficient  $g \equiv 0$  the above equations (2.5)-(2.6) reduces to (Buckdahn et al., [5] equations (2.7) and (2.10)).

Since the derivatives  $f_x, f_{xx}, f_y, \sigma_x, \sigma_{xx}, \sigma_y, \ell_x, \ell_y, g_x, g_{xx}, h_x$ , and  $h_y$  are bounded, by assumptions (H1) and (H2), we have the following estimate

$$\mathbb{E} \left[ \sup_{s \leq t \leq T} |\Psi(t)|^2 + \int_s^T |K(t)|^2 dt + \int_s^T \int_{\Theta} |\gamma_t(\theta)|^2 \mu(d\theta) dt \right] \leq C. \quad (2.7)$$

$$\mathbb{E} \left[ \sup_{s \leq t \leq T} |Q(t)|^2 + \int_s^T |R(t)|^2 dt + \int_s^T \int_{\Theta} |\Gamma_t(\theta)|^2 \mu(d\theta) dt \right] \leq C. \quad (2.8)$$

Related with  $(\Psi^*(t), K^*(t), \gamma_t^*(\theta))$  we denote

$$\begin{aligned} \delta H(t) &= \Psi^*(t) \delta f(t) + K^*(t) \delta \sigma(t) + \int_{\Theta} \delta g(t, \theta) \gamma_t^*(\theta) \mu(d\theta) - \delta \ell(t), \\ H_x(t) &= f_x(t) \Psi^*(t) + \sigma_x(t) K^*(t) + \int_{\Theta} g_x(t, \theta) \gamma_t^*(\theta) \mu(d\theta) - \ell_x(t), \\ H_{xx}(t) &= f_{xx}(t) \Psi^*(t) + \sigma_{xx}(t) K^*(t) + \int_{\Theta} g_{xx}(t, \theta) \gamma_t^*(\theta) \mu(d\theta) - \ell_{xx}(t), \end{aligned} \quad (2.9)$$

### 3. Stochastic Maximum Principle for Optimality

In this section, we obtain a necessary conditions of optimality, where the system is described by nonlinear controlled SDEs of Mean-field type with jump processes, using spike variation method. The control domain need not be convex. The proof follows the general ideas as in Buckdahn et al., [5] and Tang et al., [32]. Note that in [5] the authors studied the Brownian case only.

The main result of this paper is stated in the following theorem.

Let  $x^*(\cdot)$  be the trajectory of the control system (1.1) corresponding to the optimal control  $u^*(\cdot)$ , and  $(\Psi^*(\cdot), K^*(\cdot), \gamma^*(\cdot))$ ,  $(Q^*(\cdot), R^*(\cdot), \Gamma^*(\cdot))$  be the solution of adjoint equations (2.5) and (2.6) respectively, corresponding to  $u^*(\cdot)$ .

**Theorem 3.1.** (*Stochastic Maximum Principle for Optimality*). Let Hypotheses (H1) and (H2) hold. If  $(u^*(\cdot), x^*(\cdot))$  is an optimal solution of the control problem (1.1)-(1.2). Then there are two triple of  $\mathcal{F}_t$ -adapted processes  $(\Psi^*(\cdot), K^*(\cdot), \gamma^*(\cdot))$  and  $(Q^*(\cdot), R^*(\cdot), \Gamma^*(\cdot))$  that satisfy (2.5) and (2.6) respectively, such that for all  $u \in \mathbb{A}$  :

$$\begin{aligned} & H(t, x^*(t), \mathbb{E}(x^*(t)), u, \Psi^*(t), K^*(t), \gamma_t^*(\theta)) \\ & - H(t, x^*(t), \mathbb{E}(x^*(t)), u^*(t), \Psi^*(t), K^*(t), \gamma_t^*(\theta)) \\ & + \frac{1}{2} (\sigma(t, x^*(t), \mathbb{E}(x^*(t)), u) - \sigma(t, x^*(t), \mathbb{E}(x^*(t)), u^*(t)))^2 Q^*(t) \\ & + \frac{1}{2} \int_{\Theta} (g(t, x^*(t), u, \theta) - g(t, x^*(t), u^*(t), \theta))^2 (Q^*(t) + \Gamma_t^*(\theta)) \mu(d\theta) \leq 0. \end{aligned} \quad (3.1)$$

$$\mathbb{P}\text{-a.s., a.e. } t \in [s, T].$$

To prove *Theorem 3.1* we need some preliminary results given in the following Lemmas.

Let  $(u^*(\cdot), x^*(\cdot))$  be the optimal solution of the control problem (1.1)-(1.2). Following Tang et al., [32], and Buckdahn [5], we derive the variational inequality (3.1) in several steps, from the fact that

$$J^{s,\zeta}(u^\varepsilon(\cdot)) - J^{s,\zeta}(u^*(\cdot)) \geq 0, \quad (3.2)$$

where  $u^\varepsilon(\cdot)$  is the so called spike variation of  $u^*(\cdot)$  defined as follows.

For  $\varepsilon > 0$ , we choose a Borel measurable set  $\mathcal{E}_\varepsilon \subset [s, T]$  such that  $v(\mathcal{E}_\varepsilon) = \varepsilon$ , where  $v(\mathcal{E}_\varepsilon)$  denote the Lebesgue measure of the subset  $\mathcal{E}_\varepsilon$ , and we consider the control process which is the spike variation of  $u^*(\cdot)$

$$u^\varepsilon(t) = \begin{cases} u : t \in \mathcal{E}_\varepsilon, \\ u^*(t) : t \in [s, T] \setminus \mathcal{E}_\varepsilon, \end{cases} \quad (3.3)$$

where  $\varepsilon > 0$  is sufficiently small and  $u$  is an arbitrary element  $\mathcal{F}_t$ -measurable random variable with values in  $\mathbb{A}$ , such that  $\sup_{w \in \Omega} |u(w)| < \infty$ , which we consider as fixed from now on.

Let  $x_1^\varepsilon(\cdot)$  and  $x_2^\varepsilon(\cdot)$  be the solutions of the following SDEs respectively

$$\begin{cases} dx_1^\varepsilon(t) = \{f_x(t)x_1^\varepsilon(t) + f_y(t)\mathbb{E}(x_1^\varepsilon(t)) + \delta f(t)\mathbf{I}_{\mathcal{E}_\varepsilon}(t)\} dt \\ \quad + \{\sigma_x(t)x_1^\varepsilon(t) + \sigma_y(t)\mathbb{E}(x_1^\varepsilon(t)) + \delta \sigma(t)\mathbf{I}_{\mathcal{E}_\varepsilon}(t)\} dW(t) \\ \quad + \int_{\Theta} \{g_x(t_-, \theta)x_1^\varepsilon(t) + \delta g(t_-, \theta)\mathbf{I}_{\mathcal{E}_\varepsilon}(t)\} N(d\theta, dt), \\ x_1^\varepsilon(s) = 0, \end{cases} \quad (3.4)$$

and

$$\begin{cases} dx_2^\varepsilon(t) = \{f_x(t)x_2^\varepsilon(t) + f_y(t)\mathbb{E}(x_2^\varepsilon(t)) + \mathcal{L}_t(f, x_1^\varepsilon) + \delta f_x(t)\mathbf{I}_{\mathcal{E}_\varepsilon}(t)\} dt \\ \quad + \{\sigma_x(t)x_2^\varepsilon(t) + \sigma_y(t)\mathbb{E}(x_2^\varepsilon(t)) + \mathcal{L}_t(\sigma, x_1^\varepsilon) + \delta\sigma_x(t)\mathbf{I}_{\mathcal{E}_\varepsilon}(t)\} dW(t) \\ \quad + \int_{\Theta} \{g_x(t-, \theta)x_2^\varepsilon(t) + \mathcal{L}_{t,\theta}(g, x_1^\varepsilon) + \delta g_x(t-, \theta)\mathbf{I}_{\mathcal{E}_\varepsilon}(t)\} N(d\theta, dt), \\ x_2^\varepsilon(s) = 0. \end{cases} \quad (3.5)$$

Noting that equation (3.4) is called the first-order variational equation and equation (3.5) is called the second-order variational equation.

Our first Lemma below deals with the duality relations between  $\Psi(t)$ ,  $x_1^\varepsilon(t)$  and  $x_2^\varepsilon(t)$ .

**Lemma 3.1.** We have

$$\begin{aligned} \mathbb{E}(\Psi(T)x_1^\varepsilon(T)) &= \mathbb{E} \int_s^T x_1^\varepsilon(t) [(\ell_x(t) + \mathbb{E}(\ell_y(t)))] dt \\ &\quad + \mathbb{E} \int_s^T \{\Psi(t)\delta f(t) + K(t)\delta\sigma(t)\} \mathbf{I}_{\mathcal{E}_\varepsilon}(t) dt \\ &\quad + \mathbb{E} \int_s^T \int_{\Theta} \gamma_t(\theta)\delta g(t, \theta)\mathbf{I}_{\mathcal{E}_\varepsilon}(t)\mu(d\theta) dt, \end{aligned} \quad (3.6)$$

and

$$\begin{aligned} \mathbb{E}(\Psi(T)x_2^\varepsilon(T)) &= \mathbb{E} \int_s^T x_2^\varepsilon(t) [(\ell_x(t) + \mathbb{E}(\ell_y(t)))] dt \\ &\quad + \mathbb{E} \int_s^T \{\Psi(t)\delta f_x(t) + K(t)\delta\sigma_x(t)\} x_1^\varepsilon(t)\mathbf{I}_{\mathcal{E}_\varepsilon}(t) dt \\ &\quad + \mathbb{E} \int_s^T \int_{\Theta} \gamma_t(\theta)\delta g_x(t, \theta)x_1^\varepsilon(t)\mathbf{I}_{\mathcal{E}_\varepsilon}(t)\mu(d\theta) dt \\ &\quad + \mathbb{E} \int_s^T \Psi(t)\mathcal{L}_t(f, x_1^\varepsilon) + K(t)\mathcal{L}_t(\sigma, x_1^\varepsilon) dt \\ &\quad + \mathbb{E} \int_s^T \int_{\Theta} \gamma_t(\theta)\mathcal{L}_{t,\theta}(g, x_1^\varepsilon)\mu(d\theta) dt. \end{aligned} \quad (3.7)$$

**Proof.** By applying Itô's formula for jump processes (see Lemma A1), then we get

$$\begin{aligned} \mathbb{E}(\Psi(T)x_1^\varepsilon(T)) &= \mathbb{E} \int_s^T \Psi(t)dx_1^\varepsilon(t) + \mathbb{E} \int_s^T x_1^\varepsilon(t)d\Psi(t) \\ &\quad + \mathbb{E} \int_s^T K(t) [\sigma_x(t)x_1^\varepsilon(t) + \sigma_y(t)\mathbb{E}(x_1^\varepsilon(t)) + \delta\sigma(t)\mathbf{I}_{\mathcal{E}_\varepsilon}(t)] dt \\ &\quad + \mathbb{E} \int_s^T \int_{\Theta} \gamma_t(\theta) [g_x(t, \theta)x_1^\varepsilon(t) + \delta g(t, \theta)\mathbf{I}_{\mathcal{E}_\varepsilon}(t)] \mu(d\theta) dt \\ &= I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon + I_4^\varepsilon. \end{aligned} \quad (3.8)$$

A simple computation shows that

$$\begin{aligned} I_1^\varepsilon &= \mathbb{E} \int_s^T \Psi(t) dx_1^\varepsilon(t) \\ &= \mathbb{E} \int_s^T \{ \Psi(t) f_x(t) x_1^\varepsilon(t) + \Psi(t) f_y(t) \mathbb{E}(x_1^\varepsilon(t)) + \Psi(t) \delta f(t) \mathbf{I}_{\mathcal{E}_\varepsilon}(t) \} dt, \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} I_2^\varepsilon &= \mathbb{E} \int_s^T x_1^\varepsilon(t) d\Psi(t) \\ &= -\mathbb{E} \int_s^T \{ x_1^\varepsilon(t) f_x(t) \Psi(t) + x_1^\varepsilon(t) \mathbb{E}(f_y^\top(t) \Psi(t)) + x_1^\varepsilon(t) \sigma_x(t) K(t) \\ &\quad + x_1^\varepsilon(t) \mathbb{E}(\sigma_y(t) K(t)) + x_1^\varepsilon(t) \ell_x(t) + x_1^\varepsilon(t) \mathbb{E}(\ell_y(t)) \} dt \\ &\quad - \mathbb{E} \int_s^T \int_{\Theta} x_1^\varepsilon(t) g_x(t, \theta) \gamma_t(\theta) \mu(d\theta) dt. \end{aligned} \quad (3.10)$$

By standard arguments we get

$$\begin{aligned} I_3^\varepsilon &= \mathbb{E} \int_s^T K(t) [\sigma_x(t) x_1^\varepsilon(t) + \sigma_y(t) \mathbb{E}(x_1^\varepsilon(t)) + \delta \sigma(t) \mathbf{I}_{\mathcal{E}_\varepsilon}(t)] dt \\ &= \mathbb{E} \int_s^T K(t) \sigma_x(t) x_1^\varepsilon(t) dt + \mathbb{E} \int_s^T K(t) \sigma_y(t) \mathbb{E}(x_1^\varepsilon(t)) dt \\ &\quad + \mathbb{E} \int_s^T K(t) \delta \sigma(t) \mathbf{I}_{\mathcal{E}_\varepsilon}(t) dt, \end{aligned} \quad (3.11)$$

and

$$\begin{aligned} I_4^\varepsilon &= \mathbb{E} \int_s^T \int_{\Theta} \gamma_t(\theta) [g_x(t, \theta) x_1^\varepsilon(t) + \delta g(t, \theta) \mathbf{I}_{\mathcal{E}_\varepsilon}(t)] \mu(d\theta) dt \\ &= \mathbb{E} \int_s^T \int_{\Theta} \gamma_t(\theta) g_x(t, \theta) x_1^\varepsilon(t) \mu(d\theta) dt \\ &\quad + \mathbb{E} \int_s^T \int_{\Theta} \gamma_t(\theta) \delta g(t, \theta) \mathbf{I}_{\mathcal{E}_\varepsilon}(t) \mu(d\theta) dt. \end{aligned} \quad (3.12)$$

Finally the duality relation (3.6) follows from combining (3.9)~(3.12) and (3.8). Similarly we can prove second duality relation (3.7).

To this end we need the following estimations. Let  $x^\varepsilon(\cdot)$  be the solutions of the SDEs-(1.1) corresponding to the control  $u^\varepsilon(\cdot)$ .

**Lemma 3.2.** Let Hypotheses (H1) and (H2) hold. Then we have for any  $k \geq 1$  :

$$\mathbb{E} \left( \sup_{s \leq t \leq T} |x_1^\varepsilon(t)|^{2k} \right) \leq C \varepsilon^k. \quad (3.13)$$

$$\sup_{s \leq t \leq T} |\mathbb{E}(x_1^\varepsilon(t))|^2 \leq \varepsilon \rho(\varepsilon). \quad (3.14)$$

$$\mathbb{E}(\sup_{s \leq t \leq T} |x_2^\varepsilon(t)|^{2k}) \leq C\varepsilon^{2k}. \quad (3.15)$$

$$\mathbb{E}(\sup_{s \leq t \leq T} |x^\varepsilon(t) - x^*(t)|^{2k}) \leq C\varepsilon^k. \quad (3.16)$$

$$\mathbb{E}(\sup_{s \leq t \leq T} |x^\varepsilon(t) - x^*(t) - x_1^\varepsilon(t)|^{2k}) \leq C\varepsilon^{2k}. \quad (3.17)$$

$$\mathbb{E}(\sup_{s \leq t \leq T} |x^\varepsilon(t) - x^*(t) - x_1^\varepsilon(t) - x_2^\varepsilon(t)|^{2k}) \leq C_{k,\mu(\Theta)}\varepsilon^{2k}\rho_k(\varepsilon), \quad (3.18)$$

where  $C_k$  is a positive constant depend to  $k$  and  $\rho, \rho_k : (0, \infty) \rightarrow (0, \infty)$  such that  $\rho(\varepsilon) \rightarrow 0$  and  $\rho_k(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

To prove Lemma 3.2 we need some results given in the following Lemma.

**Lemma 3.3.** For any progressively measurable process  $(\Phi(t))_{t \in [s, T]}$  for which for any  $p > 1$ , there exists a positive constant  $C_p$  such that

$$\mathbb{E}(\sup_{s \leq t \leq T} |\Phi(t)|^p) \leq C_p. \quad (3.19)$$

Then there exists a function  $\tilde{\rho} : (0, \infty) \rightarrow (0, \infty)$  satisfying  $\tilde{\rho}(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  such that for  $\varepsilon > 0$  :

$$|\mathbb{E}(\Phi(T)x_1^\varepsilon(T))|^2 + \int_s^T |\mathbb{E}(\Phi(t)x_1^\varepsilon(t))|^2 dt \leq C_{(T,\mu(\Theta))}\varepsilon\tilde{\rho}(\varepsilon). \quad (3.20)$$

**Proof.** First we set for  $t \in [s, T] : \eta(t) = \exp\{Z(t)\}$ , where

$$\begin{aligned} Z(t) = & - \int_s^t \left[ f_x(r) - \frac{1}{2} |\sigma_x(r)|^2 - \frac{1}{2} \int_{\Theta} (g_x(r, \theta))^2 \mu(d\theta) \right] dr - \int_s^t \sigma_x(r) dw(r) \\ & - \int_s^t \int_{\Theta} g_x(r_-, \theta) N(d\theta, dr), \end{aligned}$$

and we denote by  $\rho(t) = \eta(t)^{-1} = \exp\{-Z(t)\}$ .

By using Itô formula for the exponential  $\exp\{Z(t)\}$  we get

$$d(\exp\{Z(t)\}) = \exp\{Z(t)\} dZ(t) + \frac{1}{2} \exp\{Z(t)\} d\langle Z(t); Z(t) \rangle,$$

this shows that

$$\begin{aligned} d\eta(t) &= d(\exp\{Z(t)\}) \\ &= -\eta(t) \left\{ \left[ f_x(t) - (\sigma_x(t))^2 - \int_{\Theta} (g_x(t, \theta))^2 \mu(d\theta) \right] dt \right. \\ &\quad \left. + \sigma_x(t) dW(t) + \int_{\Theta} g_x(t_-, \theta) N(d\theta, dt) \right\}. \end{aligned} \quad (3.21)$$

By applying Integration by parts formula for jumps processes  $\eta(t) x_1^\varepsilon(t)$  we have

$$\begin{aligned} d(\eta(t) x_1^\varepsilon(t)) &= \eta(t) dx_1^\varepsilon(t) + x_1^\varepsilon(t) d\eta(t) + d\langle \eta(t), x_1^\varepsilon(t) \rangle, \\ &= \mathcal{I}_1^\varepsilon + \mathcal{I}_2^\varepsilon + \mathcal{I}_3^\varepsilon. \end{aligned}$$

From (3.4) we get

$$\begin{aligned} \mathcal{I}_1^\varepsilon &= \eta(t) dx_1^\varepsilon(t) \\ &= \eta(t) \{ [f_x(t)x_1^\varepsilon(t) + f_y(t)\mathbb{E}(x_1^\varepsilon(t)) + \delta f(t)\mathbf{I}_{\mathcal{E}_\varepsilon}(t)] dt \\ &\quad + [\sigma_x(t)x_1^\varepsilon(t) + \sigma_y(t)\mathbb{E}(x_1^\varepsilon(t)) + \delta\sigma(t)\mathbf{I}_{\mathcal{E}_\varepsilon}(t)] dW(t) \\ &\quad + \int_{\Theta} \{g_x(t_-, \theta) x_1^\varepsilon(t) + \delta g(t_-, \theta)\mathbf{I}_{\mathcal{E}_\varepsilon}(t)\} N(d\theta, dt) \}. \end{aligned}$$

By using (3.21) we obtain

$$\begin{aligned} \mathcal{I}_2^\varepsilon &= x_1^\varepsilon(t) d\eta(t) \\ &= -\eta(t) f_x(t) x_1^\varepsilon(t) dt - \eta(t) \sigma_x(t) x_1^\varepsilon(t) dW(t) - \eta(t) x_1^\varepsilon(t) \int_{\Theta} g_x(t_-, \theta) N(d\theta, dt) \\ &\quad + \eta(t) x_1^\varepsilon(t) (\sigma_x(t))^2 dt + \eta(t) \int_{\Theta} (g_x(t_-, \theta))^2 x_1^\varepsilon(t) \mu(d\theta), \end{aligned}$$

and a simple computation we get

$$\begin{aligned} \mathcal{I}_3^\varepsilon &= d\langle \eta(t), x_1^\varepsilon(t) \rangle = -\eta(t) \sigma_x(t) [\sigma_x(t) x_1^\varepsilon(t) + \sigma_y(t)\mathbb{E}(x_1^\varepsilon(t)) + \delta\sigma(t)\mathbf{I}_{\mathcal{E}_\varepsilon}(t)] dt \\ &\quad - \int_{\Theta} \eta(t) g_x(t, \theta) \{g_x(t, \theta) x_1^\varepsilon(t) + \delta g(t, \theta)\mathbf{I}_{\mathcal{E}_\varepsilon}(t)\} \mu(d\theta) dt. \end{aligned}$$

Consequently, from the above equations we deduce that

$$\begin{aligned} d(\eta(t) x_1^\varepsilon(t)) &= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 \\ &= \eta(t) \{ [f_y(t)\mathbb{E}(x_1^\varepsilon(t)) + \delta f(t)\mathbf{I}_{\mathcal{E}_\varepsilon}(t)] dt \\ &\quad + [\sigma_y(t)\mathbb{E}(x_1^\varepsilon(t)) + \delta\sigma(t)\mathbf{I}_{\mathcal{E}_\varepsilon}(t)] dW(t) \\ &\quad + \int_{\Theta} \{ \delta g(t_-, \theta)\mathbf{I}_{\mathcal{E}_\varepsilon}(t) \} N(d\theta, dt) \} \\ &\quad - \eta(t) \{ \sigma_x(t) [\sigma_y(t)\mathbb{E}(x_1^\varepsilon(t)) + \delta\sigma(t)\mathbf{I}_{\mathcal{E}_\varepsilon}(t)] \\ &\quad + \int_{\Theta} g_x(t, \theta) \delta g(t, \theta)\mathbf{I}_{\mathcal{E}_\varepsilon}(t) \mu(d\theta) \} dt, \end{aligned}$$

by integrating the above equation and the fact  $\rho(t) = \eta(t)^{-1}$  we obtain

$$\begin{aligned}
x_1^\varepsilon(t) &= \rho(t) \int_s^t \eta(r) \{ [f_y(r) \mathbb{E}(x_1^\varepsilon(r)) + \delta f(t) \mathbf{I}_{\mathcal{E}_\varepsilon}(r)] \\
&\quad - \sigma_x(r) \sigma_y(r) \mathbb{E}(x_1^\varepsilon(r)) + \sigma_x(r) \delta \sigma(r) \mathbf{I}_{\mathcal{E}_\varepsilon}(r) \\
&\quad - \int_\Theta g_x(r, \theta) \delta g(r, \theta) \mathbf{I}_{\mathcal{E}_\varepsilon}(r) \mu(d\theta) \} dr \\
&\quad + \rho(t) \int_s^t \eta(r) [\sigma_y(r) \mathbb{E}(x_1^\varepsilon(r)) + \delta \sigma(r) \mathbf{I}_{\mathcal{E}_\varepsilon}(r)] dW(r) \\
&\quad + \rho(t) \int_s^t \int_\Theta \eta(r) \delta g(r, \theta) \mathbf{I}_{\mathcal{E}_\varepsilon}(r) N(d\theta, dr).
\end{aligned} \tag{3.22}$$

Since  $f_x, \sigma_x, g_x(\cdot, \theta)$  are bounded, then by using (Proposition A1, Appendix) we get: for all  $p > 1$  there exists a positive constant  $C = C_{(T, p, \mu(\Theta))}$  such that

$$\mathbb{E} \left[ \sup_{s \leq t \leq T} \left| \int_s^t \int_\Theta g_x(r, \theta) N(d\theta, dr) \right|^p \right] \leq C_{(T, p, \mu(\Theta))} \mathbb{E} \left[ \int_s^T \int_\Theta |g_x(r, \theta)|^p \mu(d\theta) dr \right],$$

which shows that

$$\mathbb{E} \left[ \sup_{s \leq t \leq T} (|\eta(t)|^p + |\rho(t)|^p) \right] \leq C_{(T, p, \mu(\Theta))}. \tag{3.23}$$

Moreover, it follows from (3.19) that

$$\mathbb{E} \left[ \sup_{s \leq t \leq T} |\Phi(t) \rho(t)|^p \right] \leq C_{(T, p, \mu(\Theta))}. \tag{3.24}$$

Next, since  $\mathcal{F}_t = (\mathcal{F}_t^{(W, N)})_{t \in [s, T]}$  then by applying *Martingale Representation Theorem* for jump processes (see Lemma A2), there exists a unique  $\gamma_t(\cdot) \in \mathbb{L}_{\mathcal{F}}^2([s, t])$  and unique  $\xi_t(\cdot, \theta) \in \mathbb{M}_{\mathcal{F}}^2([s, t])$  such that  $\forall t \in [s, T]$ :

$$\Phi(t) \rho(t) = \mathbb{E}(\Phi(t) \rho(t)) + \int_s^t \gamma_t(r) dW(r) + \int_s^t \int_\Theta \xi_t(r, \theta) N(d\theta, dr). \quad \mathbb{P}\text{-a.s.} \tag{3.25}$$

Noting that, for every  $p > 1$ , with the help of (3.22) it follows from the Bulkholder-Davis-Gundy inequality and Proposition A1 that there exists a constant  $C_{(T, p, \mu(\Theta))}$  such that: for

$p > 1$ ,

$$\begin{aligned}
& \mathbb{E} \left[ \left( \int_s^t |\gamma_t(r)|^2 dr \right)^{\frac{p}{2}} \right] + \mathbb{E} \left[ \left( \int_s^t \int_{\Theta} |\xi_t(r, \theta)|^2 \mu(d\theta) dr \right)^{\frac{p}{2}} \right] \\
\leq & C_p \mathbb{E} \left[ \sup_{s \leq \tau \leq t} \left| \int_s^{\tau} \gamma_t(r) dW(r) \right|^p \right] \\
& + C_{(T, p, \mu(\Theta))} \mathbb{E} \left[ \sup_{s \leq \tau \leq t} \left| \int_s^{\tau} \int_{\Theta} \xi_t(r, \theta) N(d\theta, dr) \right|^p \right] \\
\leq & C_p \left( 1 + \frac{1}{p-1} \right)^p \mathbb{E} \left[ \left| \int_s^t \gamma_t(r) dW(r) \right|^p \right] \\
& + C_{(T, p, \mu(\Theta))} \mathbb{E} \left[ \left| \int_s^t \int_{\Theta} \xi_t(r, \theta) N(d\theta, dr) \right|^p \right] \\
\leq & C_{(T, p, \mu(\Theta))} \mathbb{E} [|\Phi(t) \rho(t) - \mathbb{E}(\Phi(t) \rho(t))|^p] \\
\leq & C_{(T, p, \mu(\Theta))} \{ \mathbb{E}(|\Phi(t) \rho(t)|^p) + |\mathbb{E}(\Phi(t) \rho(t))|^p \} \\
\leq & C_{(T, p, \mu(\Theta))} \mathbb{E} [|\Phi(t) \rho(t)|^p] \\
\leq & C_{(T, p, \mu(\Theta))} \mathbb{E} \left[ \sup_{s \leq t \leq T} |\Phi(t) \rho(t)|^p \right] \leq C_{(T, p, \mu(\Theta))}.
\end{aligned}$$

This shows that

$$\sup_{s \leq t \leq T} \mathbb{E} \left[ \left( \int_s^t |\gamma_t(r)|^2 dr \right)^{\frac{p}{2}} \right] \leq C_{(T, p, \mu(\Theta))}, \quad (3.26)$$

and

$$\sup_{s \leq t \leq T} \mathbb{E} \left[ \left( \int_s^t \int_{\Theta} |\xi_t(r, \theta)|^2 \mu(d\theta) dr \right)^{\frac{p}{2}} \right] \leq C_{(T, p, \mu(\Theta))}. \quad (3.27)$$

Now we consider

$$\Phi(t) x_1^\varepsilon(t) = \mathcal{J}_1^\varepsilon(t) + \mathcal{J}_2^\varepsilon(t) + \mathcal{J}_3^\varepsilon(t), \quad t \in [s, T], \quad (3.28)$$

where

$$\begin{aligned}
\mathcal{J}_1^\varepsilon(t) = & \Phi(t) \rho(t) \int_s^t \eta(r) \{ [f_y(r) \mathbb{E}(x_1^\varepsilon(r)) + \delta f(t) \mathbf{I}_{\mathcal{E}_\varepsilon}(r)] \\
& - \sigma_x(r) \sigma_y(r) \mathbb{E}(x_1^\varepsilon(r)) + \sigma_x(r) \delta \sigma(r) \mathbf{I}_{\mathcal{E}_\varepsilon}(r) \\
& - \int_{\Theta} g_x(r, \theta) \delta g(r, \theta) \mathbf{I}_{\mathcal{E}_\varepsilon}(r) \mu(d\theta) \} dr,
\end{aligned}$$

$$\mathcal{J}_2^\varepsilon(t) = \Phi(t) \rho(t) \int_s^t \eta(r) [\sigma_y(r) \mathbb{E}(x_1^\varepsilon(r)) + \delta \sigma(r) \mathbf{I}_{\mathcal{E}_\varepsilon}(r)] dW(r),$$

and

$$\mathcal{J}_3^\varepsilon(t) = \Phi(t) \rho(t) \int_s^t \int_{\Theta} \eta(r) \delta g(r, \theta) \mathbf{I}_{\mathcal{E}_\varepsilon}(r) N(d\theta, dr).$$

We estimate now the first term in the right-hand side of (3.28). First, since  $f_y, \sigma_x, \sigma_y$ , are bounded and fact that  $\sup_{\theta \in \Theta} |g_x(t, \theta)| < +\infty$  (see H2) we get

$$\begin{aligned} |\mathbb{E}(\mathcal{J}_1^\varepsilon(t))| &= \left| \mathbb{E} \left\{ \Phi(t) \rho(t) \int_s^t \eta(r) [(f_y(r) - \sigma_x(r)\sigma_y(r)) \mathbb{E}(x_1^\varepsilon(r)) \right. \right. \\ &\quad \left. \left. + (\delta f(t) + \sigma_x(r)\delta\sigma(r)) \mathbf{I}_{\mathcal{E}_\varepsilon}(r) \right. \right. \\ &\quad \left. \left. - \int_\Theta g_x(r-, \theta) \delta g(r, \theta) \mathbf{I}_{\mathcal{E}_\varepsilon}(r) \mu(d\theta) \right] dr \right\} \right| \\ &\leq C_{(\mu(\Theta))} \mathbb{E} \left\{ \sup_{t \in [s, T]} |\Phi(t) \rho(t)| \sup_{t \in [s, T]} |\eta(t)| \left[ \int_s^t |\mathbb{E}(x_1^\varepsilon(r))| dr + \varepsilon \right] \right\}, \end{aligned}$$

applying Cauchy-Schwarz inequality, then from (3.23) and (3.24) (with  $p = 2$ ), we get

$$\begin{aligned} |\mathbb{E}(\mathcal{J}_1^\varepsilon(t))| &\leq C_{(\mu(\Theta))} \left[ \mathbb{E} \left( \sup_{t \in [s, T]} |\Phi(t) \rho(t)|^2 \right) \right]^{\frac{1}{2}} \\ &\quad \times \left[ \mathbb{E} \left( \sup_{t \in [s, T]} |\eta(t)|^2 \right) \right]^{\frac{1}{2}} \left[ \int_s^t |\mathbb{E}(x_1^\varepsilon(r))| dr + \varepsilon \right] \\ &\leq C_{(T, \mu(\Theta))} \left[ \int_s^t |\mathbb{E}(x_1^\varepsilon(r))| dr + \varepsilon \right], \end{aligned}$$

by applying Cauchy-Schwarz inequality and the fact that  $(a + b)^2 \leq 2a^2 + 2b^2$  we can shows that

$$\begin{aligned} |\mathbb{E}(\mathcal{J}_1^\varepsilon(t))|^2 &\leq C_{(T, \mu(\Theta))} \left[ 2 \left( \int_s^t |\mathbb{E}(x_1^\varepsilon(r))| dr \right)^2 + 2\varepsilon^2 \right] \\ &\leq C_{(T, \mu(\Theta))} \left[ \int_s^t |\mathbb{E}(x_1^\varepsilon(r))|^2 dr + \varepsilon^2 \right]. \end{aligned} \tag{3.29}$$

Next, we proceed to estimate the second term  $\mathcal{J}_2^\varepsilon(t)$ . With the help of (3.25) and the *Itô Isometry* we can get

$$\begin{aligned} \mathbb{E}(\mathcal{J}_2^\varepsilon(t)) &= \mathbb{E} \left\{ \Phi(t) \rho(t) \int_s^t \eta(r) [\sigma_y(r) \mathbb{E}(x_1^\varepsilon(r)) + \delta\sigma(r) \mathbf{I}_{\mathcal{E}_\varepsilon}(r)] dW(r) \right\} \\ &= \mathbb{E} \left\{ \left[ \mathbb{E}(\Phi(t) \rho(t)) + \int_s^t \gamma_t(r) dW(r) + \int_s^t \int_\Theta \xi_t(r-, \theta) N(d\theta, dr) \right] \right. \\ &\quad \left. \times \int_s^t \eta(r) [\sigma_y(r) \mathbb{E}(x_1^\varepsilon(r)) + \delta\sigma(r) \mathbf{I}_{\mathcal{E}_\varepsilon}(r)] dW(r) \right\} \\ &= \mathbb{E} \left[ \int_s^t \gamma_t(r) \eta(r) \sigma_y(r) \mathbb{E}(x_1^\varepsilon(r)) dr \right] + \mathbb{E} \left[ \int_s^t \gamma_t(r) \eta(r) \delta\sigma(r) \mathbf{I}_{\mathcal{E}_\varepsilon}(r) dr \right]. \end{aligned}$$

We estimate now the first term in the right hand side of the above equality. Applying (3.23)-(3.26) then we can get immediately

$$\left| \mathbb{E} \int_s^t \gamma_t(r) \eta(r) \sigma_y(r) \mathbb{E}(x_1^\varepsilon(r)) dr \right|^2 \leq C \int_s^t |\mathbb{E}(x_1^\varepsilon(r))|^2 dr, \quad (3.30)$$

however, the second term satisfies

$$\int_s^T \left\{ \left| \mathbb{E} \left[ \int_s^t \gamma_t(r) \eta(r) \delta\sigma(r) \mathbf{I}_{\mathcal{E}_\varepsilon}(r) dr \right] \right|^2 \right\} dt \leq C\varepsilon \rho_1(\varepsilon), \quad (3.31)$$

where

$$\rho_1(\varepsilon) = \left\{ \mathbb{E} \left[ \left( \int_s^T \int_s^t |\gamma_t(r)|^2 \mathbf{I}_{\mathcal{E}_\varepsilon}(t) dr dt \right)^2 \right] \right\}^{\frac{1}{2}}.$$

Noting that since  $\lim_{\varepsilon \rightarrow 0} \mathbf{I}_{\mathcal{E}_\varepsilon}(t) = 0$  in measure  $dt d\mathbb{P}$  then the by *Dominate Convergence Theorem* we get that  $\lim_{\varepsilon \rightarrow 0} \rho_1(\varepsilon) = 0$ .

Let us turn to estimate the third term  $\mathcal{J}_3^\varepsilon(t)$ . Then by Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |\mathbb{E}(\mathcal{J}_3^\varepsilon(t))|^2 &= \left| \mathbb{E} \left\{ \Phi(t) \rho(t) \int_s^t \int_{\Theta} \eta(r) \delta g(r_-, \theta) \mathbf{I}_{\mathcal{E}_\varepsilon}(r) N(d\theta, dr) \right\} \right|^2 \\ &\leq \left| \mathbb{E} \left\{ \sup_{t \in [s, T]} |\Phi(t) \rho(t)| \sup_{t \in [s, T]} |\eta(t)| \int_s^t \int_{\Theta} \delta g(r_-, \theta) \mathbf{I}_{\mathcal{E}_\varepsilon}(r) N(d\theta, dr) \right\} \right|^2 \\ &\leq C \mathbb{E} \left[ \sup_{t \in [s, T]} |\Phi(t) \rho(t)|^2 \right] \left[ \mathbb{E} \sup_{t \in [s, T]} |\eta(t)|^2 \right] \\ &\quad \times \mathbb{E} \left[ \left| \int_s^t \int_{\Theta} \delta g(r_-, \theta) \mathbf{I}_{\mathcal{E}_\varepsilon}(r) N(d\theta, dr) \right|^2 \right], \end{aligned}$$

by applying Propositions A1 then from (3.23) and (3.24) (with  $p = 2$ ), we get

$$\begin{aligned} |\mathbb{E}(\mathcal{J}_3^\varepsilon(t))|^2 &\leq C_{(T, \mu(\Theta))} \int_s^t \int_{\Theta} |\delta g(r, \theta) \mathbf{I}_{\mathcal{E}_\varepsilon}(r)|^2 \mu(d\theta) dr \\ &\leq C_{(T, \mu(\Theta))} \int_s^t \sup_{\theta \in \Theta} |\delta g(r, \theta)|^2 \int_{\Theta} \mathbf{I}_{\mathcal{E}_\varepsilon}(r) \mu(d\theta) dr \\ &\leq C_{(T, \mu(\Theta))} \varepsilon. \end{aligned} \quad (3.32)$$

Combining (3.30)~(3.32) and the fact that

$$|\mathbb{E}(\Phi(t) x_1^\varepsilon(t))|^2 \leq 2 |\mathbb{E}(\mathcal{J}_1^\varepsilon(t))|^2 + 4 |\mathbb{E}(\mathcal{J}_2^\varepsilon(t))|^2 + 4 |\mathbb{E}(\mathcal{J}_3^\varepsilon(t))|^2, t \in [s, T],$$

we conclude

$$\begin{aligned} |\mathbb{E}(\Phi(t) x_1^\varepsilon(t))|^2 &\leq C_{(T, \mu(\Theta))} \left[ \varepsilon^2 + \varepsilon + \left| \mathbb{E} \left[ \int_s^t \gamma_t(r) \eta(r) \delta\sigma(r) \mathbf{I}_{\mathcal{E}_\varepsilon}(r) dr \right] \right|^2 \right. \\ &\quad \left. + \int_s^t |\mathbb{E}(x_1^\varepsilon(r))|^2 dr \right], \end{aligned} \quad (3.33)$$

integrating the above inequality, then with the help of (3.31) we get

$$\int_s^t |\mathbb{E}(\Phi(r) x_1^\varepsilon(r))|^2 dr \leq C_{(T, \mu(\Theta))} \left[ \varepsilon^2 + \varepsilon + \varepsilon \rho_1(\varepsilon) + \int_s^t \int_s^e |\mathbb{E}(x_1^\varepsilon(r))|^2 dr de \right]. \quad (3.34)$$

Now, taking  $\Phi(t) = 1$  in (3.34) and from Gronwall's Lemma we have

$$\int_s^t |\mathbb{E}(x_1^\varepsilon(r))|^2 dr \leq C_{(T, \mu(\Theta))} (\varepsilon^2 + \varepsilon + \varepsilon \rho_1(\varepsilon)). \quad (3.35)$$

Consequently, from (3.34) it holds that

$$\int_s^t |\mathbb{E}(\Phi(r) x_1^\varepsilon(r))|^2 dr \leq C_{(T, \mu(\Theta))} (\varepsilon^2 + \varepsilon + \varepsilon \rho_1(\varepsilon)). \quad (3.36)$$

Furthermore, from (3.23), then by simple computation (with  $t = T$ ) we can shows that

$$\left| \mathbb{E} \left[ \int_s^T \gamma_t(r) \eta(r) \delta\sigma(r) \mathbf{I}_{\mathcal{E}_\varepsilon}(r) dr \right] \right|^2 \leq C \varepsilon \rho_T(\varepsilon), \quad (3.37)$$

where

$$\rho_T(\varepsilon) = \left\{ \mathbb{E} \left[ \left( \int_s^T |\gamma_T(r)|^2 \mathbf{I}_{\mathcal{E}_\varepsilon}(t) dr \right)^2 \right] \right\}^{\frac{1}{2}}.$$

Noting that since  $\lim_{\varepsilon \rightarrow 0} \mathbf{I}_{\mathcal{E}_\varepsilon}(t) = 0$  in measure  $dt d\mathbb{P}$  then with the help *Dominate Convergence Theorem* we can shows that  $\lim_{\varepsilon \rightarrow 0} \rho_T(\varepsilon) = 0$ .

By combining (3.33), (3.35) and (3.37) we conclude

$$|\mathbb{E}(\Phi(T) x_1^\varepsilon(T))|^2 \leq C_{(T, \mu(\Theta))} (\varepsilon + \varepsilon^2 + \varepsilon \rho_1(\varepsilon) + \varepsilon \rho_T(\varepsilon)). \quad (3.38)$$

Finally by setting  $\tilde{\rho}(\varepsilon) = (\varepsilon + \varepsilon^2 + \varepsilon \rho_1(\varepsilon) + \varepsilon \rho_T(\varepsilon)) \rightarrow 0, \varepsilon \rightarrow 0$ , then the desired result (3.20) follows immediately from (3.36) and (3.38). This completes the proof of Lemma 3.3.

### **Proof of Lemma 3.2.**

*Proof of estimate (3.14):* using (3.4) it holds that

$$\mathbb{E}(x_1^\varepsilon(t)) = \int_s^t \{ \mathbb{E}[f_x(r) x_1^\varepsilon(r)] + \mathbb{E}(f_y(r)) \mathbb{E}(x_1^\varepsilon(r)) + \mathbb{E}(\delta f(r) \mathbf{I}_{\mathcal{E}_\varepsilon}(r)) \} dr,$$

then we have

$$\begin{aligned} |\mathbb{E}(x_1^\varepsilon(t))|^2 &\leq 2 \left| \int_s^t \mathbb{E}[f_x(r) x_1^\varepsilon(r)] dr \right|^2 \\ &\quad + 2 \left| \int_s^t (\mathbb{E}(f_y(r)) \mathbb{E}(x_1^\varepsilon(r)) + \mathbb{E}(\delta f(r) \mathbf{I}_{\mathcal{E}_\varepsilon}(r))) dr \right|^2, \end{aligned} \quad (3.39)$$

by setting  $\Phi(t) = f_x(t)$  in (3.36), then by the helps of Cauchy-Schwarz inequality and fact that  $t \leq T$  we get

$$\left| \int_s^t \mathbb{E}[f_x(r) x_1^\varepsilon(r)] dr \right|^2 \leq T \int_s^t |\mathbb{E}[f_x(r) x_1^\varepsilon(r)]|^2 dr \leq C_{(T, \mu(\Theta))} (\varepsilon^2 + \varepsilon + \varepsilon \rho_1(\varepsilon)),$$

thus, in view of assumption (H1), then from (3.39) we obtain

$$|\mathbb{E}(x_1^\varepsilon(t))| \leq [C_{(T,\mu(\Theta))}(\varepsilon^2 + \varepsilon + \varepsilon\rho_1(\varepsilon))]^{\frac{1}{2}} + C \int_s^t (\varepsilon + \mathbb{E}|x_1^\varepsilon(r)|) dr.$$

Finally by applying Gronwall's Lemma, the estimate (3.14) follows with  $\rho(\varepsilon) = C_{(T,s,\mu(\Theta))}(1 + \varepsilon + \rho_1(\varepsilon))$ .

*Proof of estimate (3.18):* First we set

$$\lambda^\varepsilon(t) := x^\varepsilon(t) - x^*(t) - x_1^\varepsilon(t) - x_2^\varepsilon(t). \quad (3.40)$$

From SDEs (1.1), (3.4) and (3.5) we get

$$d\lambda^\varepsilon(t) = \Pi_f^\varepsilon(t) dt + \Pi_\sigma^\varepsilon(t) dW(t) + \int_\Theta \Lambda_g^\varepsilon(t, \theta) N(d\theta, dt), \quad (3.41)$$

where for  $\varphi = f, \sigma, \ell$

$$\begin{aligned} \Pi_\varphi^\varepsilon(t) &= \varphi(t, x^\varepsilon(t), \mathbb{E}(x^\varepsilon(t)), u^\varepsilon(t)) - \varphi(t, x^*(t), \mathbb{E}(x^*(t)), u^*(t)) \\ &\quad - \varphi_x(t)(x_1^\varepsilon(t) + x_2^\varepsilon(t)) - \{\varphi_y(t) \mathbb{E}(x_1^\varepsilon(t) + x_2^\varepsilon(t)) \\ &\quad + \mathcal{L}_t(\varphi, x_1^\varepsilon) + (\delta\varphi(t) + \delta\varphi_x(t) x_1^\varepsilon(t)) \mathbf{I}_{\mathcal{E}_\varepsilon}(t)\}, \end{aligned} \quad (3.42)$$

and

$$\begin{aligned} \Lambda_g^\varepsilon(t, \theta) &= g(t, x^\varepsilon(t_-), u^\varepsilon(t), \theta) - \{g(t, x^*(t_-), u^*(t), \theta) \\ &\quad + g_x(t, \theta)[x_1^\varepsilon(t) + x_2^\varepsilon(t)] + \mathcal{L}_{t,\theta}(g, x_1^\varepsilon) + [\delta g(t, \theta) + \delta g_x(t, \theta)] \mathbf{I}_{\mathcal{E}_\varepsilon}(t)\}. \end{aligned} \quad (3.43)$$

First we estimate the term  $\Pi_\varphi^\varepsilon(t)$ .

*Estimates of  $\Pi_\varphi^\varepsilon(t)$ :*

$$\begin{aligned} &\varphi(t, x^\varepsilon(t), \mathbb{E}(x^\varepsilon(t)), u^\varepsilon(t)) - \varphi(t, x^*(t), \mathbb{E}(x^*(t)), u^*(t)) \\ &= \int_0^1 [\varphi_x^e(t)(x^\varepsilon(t) - x^*(t)) + \varphi_y^e(t)(\mathbb{E}(x^\varepsilon(t)) - \mathbb{E}(x^*(t)))] de, \end{aligned}$$

where, for the subscript  $\varkappa$  which indicates the first and the second order derivatives of  $\varphi$ , respectively, with respect to  $\varkappa = x, xx, y, xy, yy$ , and for real  $\hbar \in [0, 1]$ :

$$\varphi_{\varkappa}^\hbar(t) = \varphi_{\varkappa}(t, x^*(t) + \hbar(x^\varepsilon(t) - x^*(t)), \mathbb{E}(x^*(t) + \hbar(x^\varepsilon(t) - x^*(t))), u^\varepsilon(t)).$$

Moreover,

$$\begin{aligned} &\varphi(t, x^\varepsilon(t), \mathbb{E}(x^\varepsilon(t)), u^\varepsilon(t)) - \varphi(t, x^*(t), \mathbb{E}(x^*(t)), u^*(t)) \\ &\quad - [\varphi_x(t)(x_1^\varepsilon(t) + x_2^\varepsilon(t)) + \varphi_y(t) \mathbb{E}(x_1^\varepsilon(t) + x_2^\varepsilon(t))] \\ &= \int_0^1 \{\varphi_x^e(t)\lambda^\varepsilon(t) + \varphi_y^e(t)\mathbb{E}(\lambda^\varepsilon(t)) + (\varphi_x^e(t) - \varphi_x(t))(x_1^\varepsilon(t) + x_2^\varepsilon(t)) \\ &\quad + (\varphi_y^e(t) - \varphi_y(t)) \mathbb{E}(x_1^\varepsilon(t) + x_2^\varepsilon(t))\} de. \end{aligned}$$

By similar arguments we get

$$\begin{aligned}
\varphi_x^\varepsilon(t) - \varphi_x(t) &= e \int_0^1 \{ \varphi_{xx}^{e,\alpha}(t) (x^\varepsilon(t) - x^*(t)) + \varphi_{xy}^{e,\alpha}(t) \mathbb{E} (x^\varepsilon(t) - x^*(t)) \} d\alpha \\
&\quad + \delta \varphi_x(t) \mathbf{I}_{\mathcal{E}_\varepsilon}(t) \\
&= e \int_0^1 \{ \varphi_{xx}^{e,\alpha}(t) \lambda^\varepsilon(t) + \varphi_{xy}^{e,\alpha}(t) \mathbb{E} (\lambda^\varepsilon(t)) \} d\alpha \\
&\quad + e \int_0^1 \{ \varphi_{xx}^{e,\alpha}(t) (x_1^\varepsilon(t) + x_2^\varepsilon(t)) \} d\alpha \\
&\quad + e \int_0^1 \{ \varphi_{xy}^{e,\alpha}(t) \mathbb{E} (x_1^\varepsilon(t) + x_2^\varepsilon(t)) \} d\alpha + \delta \varphi_x(t) \mathbf{I}_{\mathcal{E}_\varepsilon}(t),
\end{aligned}$$

and

$$\begin{aligned}
\varphi_y^\varepsilon(t) - \varphi_y(t) &= e \int_0^1 \{ \varphi_{xy}^{e,\alpha}(t) (x_1^\varepsilon(t) + x_2^\varepsilon(t)) + \varphi_{yy}^{e,\alpha}(t) \mathbb{E} (x_1^\varepsilon(t) + x_2^\varepsilon(t)) \} d\alpha \\
&\quad + e \int_0^1 \{ \varphi_{xy}^{e,\alpha}(t) \lambda^\varepsilon(t) + \varphi_{yy}^{e,\alpha}(t) \mathbb{E} (\lambda^\varepsilon(t)) \} d\alpha + \delta \varphi_y(t) \mathbf{I}_{\mathcal{E}_\varepsilon}(t).
\end{aligned}$$

Next we introduce the following notations:

$$\left\{ \begin{aligned}
Z_\varphi^{1,\varepsilon}(t) &= \int_0^1 \int_0^1 e \{ \varphi_{xx}^{e,\alpha}(t) \lambda^\varepsilon(t) (x_1^\varepsilon(t) + x_2^\varepsilon(t)) \\
&\quad + \varphi_{xy}^{e,\alpha}(t) (x_1^\varepsilon(t) + x_2^\varepsilon(t)) \mathbb{E} (\lambda^\varepsilon(t)) + \lambda^\varepsilon(t) \mathbb{E} (x_1^\varepsilon(t) + x_2^\varepsilon(t)) \} d\alpha de \\
&\quad + \int_0^1 \int_0^1 e \{ \varphi_{yy}^{e,\alpha}(t) \mathbb{E} (\lambda^\varepsilon(t)) \mathbb{E} (x_1^\varepsilon(t) + x_2^\varepsilon(t)) \} d\alpha de \\
Z_\varphi^{2,\varepsilon}(t) &= \int_0^1 \int_0^1 e \left\{ \varphi_{xx}^{e,\alpha}(t) \left[ (x_1^\varepsilon(t) + x_2^\varepsilon(t))^2 - (x_1^\varepsilon(t))^2 \right] \right. \\
&\quad + 2\varphi_{xy}^{e,\alpha}(t) (x_1^\varepsilon(t) + x_2^\varepsilon(t)) \mathbb{E} (x_1^\varepsilon(t) + x_2^\varepsilon(t)) \} d\alpha de \\
&\quad + \int_0^1 \int_0^1 e \left\{ \varphi_{yy}^{e,\alpha}(t) (\mathbb{E} (x_1^\varepsilon(t) + x_2^\varepsilon(t)))^2 \right\} d\alpha de \\
Z_\varphi^{3,\varepsilon}(t) &= \int_0^1 \int_0^1 e \left\{ \varphi_{xx}^{e,\alpha}(t) - \varphi_{xx}^{e,\alpha}(t) (x_1^\varepsilon(t))^2 \right\} d\alpha de \\
Z_\varphi^{4,\varepsilon}(t) &= [\delta \varphi_x(t) x_2^\varepsilon(t) + \delta \varphi_y(t) \mathbb{E} (x_1^\varepsilon(t) + x_2^\varepsilon(t))] \mathbf{I}_{\mathcal{E}_\varepsilon}(t).
\end{aligned} \right.$$

From (3.42) we get

$$\begin{aligned}
\Pi_\varphi^\varepsilon(t) &= Z_\varphi^{1,\varepsilon}(t) + Z_\varphi^{2,\varepsilon}(t) + Z_\varphi^{3,\varepsilon}(t) + Z_\varphi^{4,\varepsilon}(t) \\
&\quad + \int_0^1 \{ \varphi_x^\varepsilon(t) \lambda^\varepsilon(t) + \varphi_y^\varepsilon(t) \mathbb{E} (\lambda^\varepsilon(t)) \} de,
\end{aligned}$$

applying (3.40) together with estimates (3.15) and (3.17) we get  $k \geq 1$ .

$$\mathbb{E} \left[ \sup_{t \in [sT]} |\lambda^\varepsilon(t)|^{2k} \right] \leq C_k \varepsilon^{2k}. \quad (3.44)$$

Combining estimates (3.44), (3.13) and (3.15) we get

$$\mathbb{E} \left[ \sup_{t \in [s, T]} |Z_\varphi^{1, \varepsilon}(t)|^{2k} \right] \leq C_k \varepsilon^{3k}. \quad (3.45)$$

Similar arguments developed above with the helps of estimates (3.13), (3.14) and (3.15) we can prove

$$\mathbb{E} \left[ \sup_{t \in [s, T]} |Z_\varphi^{2, \varepsilon}(t)|^{2k} \right] \leq C_k \varepsilon^{2k} \rho_{1,k}(\varepsilon). \quad (3.46)$$

where  $\rho_{1,k}(\varepsilon) = (\varepsilon^k + \varepsilon^{2k} + \varepsilon^k \rho^k(\varepsilon) + \rho^k(\varepsilon)) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . From *Lebesgue's bounded convergence theorem* it holds that

$$\mathbb{E} \left[ \left( \int_s^T |Z_\varphi^{3, \varepsilon}(t)|^2 dt \right)^k \right] \leq C_k \varepsilon^{2k} \left[ \mathbb{E} \left( \int_s^T \int_0^1 \int_0^1 |\varphi_{xx}^{e, \alpha}(t) - \varphi_{xx}(t)|^{4k} d\alpha dedt \right) \right]^{\frac{1}{2}}, \quad (3.47)$$

here, if we denote  $\rho_{2,k}(\varepsilon) = \left[ \mathbb{E} \left( \int_s^T \int_0^1 \int_0^1 |\varphi_{xx}^{e, \alpha}(t) - \varphi_{xx}(t)|^{4k} d\alpha dedt \right) \right]^{\frac{1}{2}}$  then  $\lim_{\varepsilon \rightarrow 0} \rho_{2,k}(\varepsilon) = 0$ . Also,

$$\mathbb{E} \left[ \left( \int_s^T |Z_\varphi^{4, \varepsilon}(t)|^2 dt \right)^k \right] \leq C_k \varepsilon^{2k} \rho_{3,k}(\varepsilon), \quad (3.48)$$

where  $\rho_{3,k}(\varepsilon) = (\varepsilon^k + \rho^k(\varepsilon)) \rightarrow 0$ , as  $\varepsilon \rightarrow 0$ . Combining estimates (3.45)~(3.48) we deduce

$$\mathbb{E} \left[ \left( \int_s^t |\Pi_\varphi^\varepsilon(r)|^2 dr \right)^k \right] \leq C_k \varepsilon^{2k} \rho_k(\varepsilon) + C_k \left[ \int_s^t \mathbb{E} \left( |\lambda^\varepsilon(r)|^{2k} \right) dr \right], \quad (3.49)$$

where  $\rho_k(\varepsilon) = (\varepsilon^k + \rho_{1,k}(\varepsilon) + \rho_{2,k}(\varepsilon) + \rho_{3,k}(\varepsilon)) \rightarrow 0$ , as  $\varepsilon \rightarrow 0$ .

Now, let us turn to estimate the jump terms  $\Lambda_g^\varepsilon(t, \theta)$ .

*Estimates of  $\Lambda_g^\varepsilon(t, \theta)$  :* We have for  $t \in [s, T]$ ,

$$g(t, x^\varepsilon(t), u^\varepsilon(t), \theta) - \varphi(t, x^*(t), u^*(t), \theta) = \int_0^1 (g_x^e(t, \theta) (x^\varepsilon(t) - x^*(t))) de,$$

where, for the subscript  $\varkappa$  which indicates the first and the second order derivatives of  $g$ , respectively, with respect to  $\varkappa = x, xx$ , and for real  $\hbar \in [0, 1]$  :

$$g_{\varkappa}^{\hbar}(t, \theta) = g_{\varkappa}(t, x^*(t) + \hbar(x^\varepsilon(t) - x^*(t)), u^*(t), \theta).$$

Moreover,

$$\begin{aligned} & g(t, x^\varepsilon(t), u^\varepsilon(t), \theta) - g(t, x^*(t), u^*(t), \theta) - g_x(t, \theta) (x_1^\varepsilon(t) + x_2^\varepsilon(t)) \\ &= \int_0^1 \{g_x^e(t, \theta) \lambda^\varepsilon(t) + (g_x^e(t, \theta) - g_x(t, \theta)) (x_1^\varepsilon(t) + x_2^\varepsilon(t))\} de. \end{aligned}$$

By similar arguments we get

$$\begin{aligned}
g_x^\varepsilon(t, \theta) - g_x(t, \theta) &= e \int_0^1 \{g_{xx}^{e, \alpha}(t, \theta) (x^\varepsilon(t) - x^*(t)) + \mathbb{E}(x^\varepsilon(t) - x^*(t))\} d\alpha \\
&\quad + \delta g_x(t, \theta) \mathbf{I}_{\mathcal{E}_\varepsilon}(t) \\
&= e \int_0^1 \{g_{xx}^{e, \alpha}(t, \theta) \lambda^\varepsilon(t)\} d\alpha \\
&\quad + e \int_0^1 \{g_{xx}^{e, \alpha}(t, \theta) (x_1^\varepsilon(t) + x_2^\varepsilon(t))\} d\alpha + \delta g_x(t, \theta) \mathbf{I}_{\mathcal{E}_\varepsilon}(t).
\end{aligned}$$

Next we introduce the following notations:

$$\begin{cases}
Z_g^{1, \varepsilon}(t, \theta) = \int_0^1 \int_0^1 e \{g_{xx}^{e, \alpha}(t, \theta) \lambda^\varepsilon(t) (x_1^\varepsilon(t) + x_2^\varepsilon(t)) \\
\quad + \lambda^\varepsilon(t) \mathbb{E}(x_1^\varepsilon(t) + x_2^\varepsilon(t))\} d\alpha de \\
Z_g^{2, \varepsilon}(t, \theta) = \int_0^1 \int_0^1 e \left\{ g_{xx}^{e, \alpha}(t, \theta) \left[ (x_1^\varepsilon(t) + x_2^\varepsilon(t))^2 - (x_1^\varepsilon(t))^2 \right] \right. \\
Z_g^{3, \varepsilon}(t, \theta) = \int_0^1 \int_0^1 e \left\{ g_{xx}^{e, \alpha}(t, \theta) - g_{xx}^{e, \alpha}(t, \theta) (x_1^\varepsilon(t))^2 \right\} d\alpha de \\
Z_g^{4, \varepsilon}(t, \theta) = \delta g_x(t, \theta) x_2^\varepsilon(t) \mathbf{I}_{\mathcal{E}_\varepsilon}(t).
\end{cases}$$

From (3.43) we get

$$\begin{aligned}
\Lambda_g^\varepsilon(t, \theta) &= Z_\varphi^{1, \varepsilon}(t, \theta) + Z_\varphi^{2, \varepsilon}(t, \theta) + Z_\varphi^{3, \varepsilon}(t, \theta) + Z_\varphi^{4, \varepsilon}(t, \theta) \\
&\quad + \int_0^1 g_x^\varepsilon(t, \theta) \lambda^\varepsilon(t) de.
\end{aligned}$$

By applying similar arguments developed in estimate  $\Pi_\varphi^\varepsilon(t)$  we can get

$$\mathbb{E} \left[ \sup_{t \in [sT]} |\Lambda_g^\varepsilon(t, \theta)|^{2k} \right] \leq C_k \varepsilon^{2k} \rho_k(\varepsilon), \quad (3.50)$$

where  $\rho_k(\varepsilon) \rightarrow 0$ , as  $\varepsilon \rightarrow 0$ .

Finally by combining (3.49), (3.50) and (3.41) with the help of Propositions A1, and Gronwall's Lemma, we conclude

$$\mathbb{E} \left[ \sup_{t \in [sT]} |\lambda^\varepsilon(t)|^{2k} \right] \leq C_{k, \mu(\Theta)} \varepsilon^{2k} \rho_k(\varepsilon). \quad (3.51)$$

This completes the proof of estimate (3.18).

Noting that estimates (3.13), (3.15), (3.16) and (3.17), follows from standard arguments.

Now by applying estimates (3.51), (3.49) the following estimates hold.

**Corollary 3.1.** We have for  $\varphi = f, \sigma, \ell$

$$\mathbb{E} \left[ \left( \int_s^T |\Pi_\varphi^\varepsilon(r)|^2 dr \right)^k \right] \leq C_k \varepsilon^{2k} \rho_k(\varepsilon), \quad (3.52)$$

$$\mathbb{E} [|\Pi_h^\varepsilon(T)|] \leq C_k \varepsilon \rho(\varepsilon), \quad (3.53)$$

where

$$\begin{aligned} \Pi_h^\varepsilon(T) &= h(x^\varepsilon(T), \mathbb{E}(x^\varepsilon(T))) - h(x^*(T), \mathbb{E}(x^*(T))) - h_x(T)(x_1^\varepsilon(T) + x_2^\varepsilon(T)) \\ &\quad - \{h_y(T) \mathbb{E}(x_1^\varepsilon(T) + x_2^\varepsilon(T)) + \mathcal{L}_T(h, x_1^\varepsilon)\}, \end{aligned}$$

and  $\rho_k(\varepsilon)$ ,  $\rho(\varepsilon)$  tends to 0 as  $\varepsilon \rightarrow 0$ .

**Lemma 3.4.** We have

$$\begin{aligned} &\mathbb{E} [h_{xx}(x^*(T), \mathbb{E}(x^*(T))) x_1^\varepsilon(T)^2] \\ &= -\mathbb{E} \int_s^T \left\{ -H_{xx}(t)(x_1^\varepsilon(t))^2 + Q^*(t)\sigma_y^2(t)(\mathbb{E}(x_1^\varepsilon(t)))^2 + Q^*(t)(\delta\sigma(t))^2 \right. \\ &\quad + \int_\Theta (\delta g(t, \theta))^2 \mu(d\theta) \mathbf{I}_{\mathcal{E}_\varepsilon}(t) + \Gamma_t^*(\theta) \int_\Theta (\delta g(t, \theta))^2 \mathbf{I}_{\mathcal{E}_\varepsilon}(t) \mu(d\theta) \\ &\quad + 2(\mathbb{E}(x_1^\varepsilon(t))) x_1^\varepsilon(t) [Q^*(t)f_y(t) + Q^*(t)\sigma_x(t)\sigma_y(t) + R^*(t)\sigma_y(t)] \\ &\quad \left. + Q^*(t) \int_\Theta (g_x(t, \theta))^2 (x_1^\varepsilon(t))^2 \mu(d\theta) \right\} dt. \end{aligned} \quad (3.54)$$

**Proof.** By using integration by parts formula for jumps processes to  $Q^*(t)(x_1^\varepsilon(t))^2$  (see Lemma A1) and taking expectation, we get from (3.13) and (3.14)

$$\begin{aligned} \mathbb{E}(Q^*(T)x_1^\varepsilon(T)^2) &= \mathbb{E} \int_s^T Q^*(t)d((x_1^\varepsilon(t))^2) + \mathbb{E} \int_s^T (x_1^\varepsilon(t))^2 dQ^*(t) \\ &\quad + \mathbb{E} \int_s^T R^*(t)2x_1^\varepsilon(t) [\sigma_x(t)x_1^\varepsilon(t) + \sigma_y(t)\mathbb{E}(x_1^\varepsilon(t)) + \delta\sigma(t)\mathbf{I}_{\mathcal{E}_\varepsilon}(t)] dt \\ &\quad + \mathbb{E} \int_s^T \int_\Theta \Gamma_t^*(\theta)2x_1^\varepsilon(t) [g_x(t, \theta)x_1^\varepsilon(t) + \delta g(t, \theta)\mathbf{I}_{\mathcal{E}_\varepsilon}(t)] \mu(d\theta) dt \\ &= \mathcal{J}_1^\varepsilon + \mathcal{J}_2^\varepsilon + \mathcal{J}_3^\varepsilon + \mathcal{J}_4^\varepsilon. \end{aligned} \quad (3.55)$$

By using Itô formula to jump process  $(x_1^\varepsilon(t))^2$  (see Situ [31]) we have

$$\begin{aligned} \mathcal{J}_1^\varepsilon &= \mathbb{E} \int_s^T Q^*(t)d((x_1^\varepsilon(t))^2) \\ &= \mathbb{E} \int_s^T Q^*(t) \{2x_1^\varepsilon(t) [f_x(t)x_1^\varepsilon(t) + f_y(t)\mathbb{E}(x_1^\varepsilon(t)) + \delta f(t)\mathbf{I}_{\mathcal{E}_\varepsilon}(t)] \\ &\quad + \{\sigma_x(t)x_1^\varepsilon(t) + \sigma_y(t)\mathbb{E}(x_1^\varepsilon(t)) + \delta\sigma(t)\mathbf{I}_{\mathcal{E}_\varepsilon}(t)\}^2 \\ &\quad + \int_\Theta \{g_x(t, \theta)x_1^\varepsilon(t) + \delta g(t, \theta)\mathbf{I}_{\mathcal{E}_\varepsilon}(t)\}^2 \mu(d\theta) \} dt. \end{aligned} \quad (3.56)$$

Applying (2.6) we can get

$$\begin{aligned}
\mathcal{J}_2^\varepsilon &= \mathbb{E} \int_s^T (x_1^\varepsilon(t))^2 dQ^*(t) \\
&= -\mathbb{E} \int_s^T (x_1^\varepsilon(t))^2 \{2f_x(t) Q^*(t) + \sigma_x^2(t) Q^*(t) + 2\sigma_x(t) R^*(t) \\
&\quad + \int_{\Theta} (g_x(t, \theta))^2 (\Gamma_t^*(\theta) + Q^*(t)) \mu(d\theta) + 2 \int_{\Theta} \Gamma_t^*(\theta) g_x(t, \theta) \mu(d\theta) \\
&\quad + H_{xx}(t)\} dt.
\end{aligned} \tag{3.57}$$

A simple computations shows that

$$\begin{aligned}
\mathcal{J}_3^\varepsilon &= \mathbb{E} \int_s^T R^*(t) 2x_1^\varepsilon(t) [\sigma_x(t) x_1^\varepsilon(t) + \sigma_y(t) \mathbb{E}(x_1^\varepsilon(t)) + \delta\sigma(t) \mathbf{I}_{\mathcal{E}_\varepsilon}(t)] dt \\
&= 2\mathbb{E} \int_s^T \left\{ R^*(t) \sigma_x(t) (x_1^\varepsilon(t))^2 \right. \\
&\quad + R^*(t) \sigma_y(t) \mathbb{E}(x_1^\varepsilon(t)) x_1^\varepsilon(t) dt \\
&\quad \left. + R^*(t) \delta\sigma(t) x_1^\varepsilon(t) \mathbf{I}_{\mathcal{E}_\varepsilon}(t) \right\} dt,
\end{aligned} \tag{3.58}$$

and

$$\begin{aligned}
\mathcal{J}_4^\varepsilon &= 2\mathbb{E} \int_s^T \int_{\Theta} \Gamma_t^*(\theta) x_1^\varepsilon(t) [g_x(t, \theta) x_1^\varepsilon(t) + \delta g(t, \theta) \mathbf{I}_{\mathcal{E}_\varepsilon}(t)] \mu(d\theta) dt \\
&= 2\mathbb{E} \int_s^T \int_{\Theta} \Gamma_t^*(\theta) g_x(t, \theta) (x_1^\varepsilon(t))^2 \mu(d\theta) dt \\
&\quad + 2\mathbb{E} \int_s^T \int_{\Theta} \Gamma_t^*(\theta) \delta g(t, \theta) x_1^\varepsilon(t) \mathbf{I}_{\mathcal{E}_\varepsilon}(t) \mu(d\theta) dt.
\end{aligned} \tag{3.59}$$

Thus, by combining (3.56)~(3.59) together with (3.55) it follows that

$$\begin{aligned}
&\mathbb{E}(Q^*(T) (x_1^\varepsilon(T))^2) \\
&= \mathbb{E} \int_s^T \left\{ -H_{xx}(t) (x_1^\varepsilon(t))^2 \right. \\
&\quad + Q^*(t) \sigma_y^2(t) (\mathbb{E}(x_1^\varepsilon(t)))^2 + Q^*(t) \left( (\delta\sigma(t))^2 + \int_{\Theta} (\delta g(t, \theta))^2 \mu(d\theta) \right) \mathbf{I}_{\mathcal{E}_\varepsilon}(t) \\
&\quad + \int_{\Theta} \Gamma_t^*(\theta) (\delta g(t, \theta))^2 \mathbf{I}_{\mathcal{E}_\varepsilon}(t) \mu(d\theta) \\
&\quad + 2(\mathbb{E}(x_1^\varepsilon(t))) x_1^\varepsilon(t) [Q^*(t) f_y(t) + Q^*(t) \sigma_x(t) \sigma_y(t) + R^*(t) \sigma_y(t)] \\
&\quad \left. + Q^*(t) \int_{\Theta} (g_x(t, \theta))^2 (x_1^\varepsilon(t))^2 \mu(d\theta) \right\} dt.
\end{aligned}$$

Finally, since  $Q^*(T) = -h_{xx}(x^*(T), \mathbb{E}(x^*(T)))$ , this completes the proof of Lemma 3.4.

The following Lemma gives estimates related to the adjoint processes  $(\Psi^*(\cdot), K^*(\cdot), \gamma^*(\cdot))$  and  $(Q^*(\cdot), R^*(\cdot), \Gamma^*(\cdot))$  given by (2.5), (2.6) respectively.

**Lemma 3.5.** We have

$$\mathbb{E} \left\{ \int_s^T \left| \left[ \Psi^*(t) \delta f_x(t) + K^*(t) \delta \sigma_x(t) + \int_{\Theta} \gamma_t^*(\theta) \delta g_x(t, \theta) \mu(d\theta) \right] x_1^\varepsilon(t) \mathbf{I}_{\mathcal{E}_\varepsilon}(t) \right| dt \right\} \leq C\varepsilon\rho(\varepsilon), \quad (3.60)$$

$$\mathbb{E} \left\{ \int_s^T \left| [Q^*(t) f_y(t) + Q^*(t) \sigma_x(t) \sigma_y(t) + R^*(t) \sigma_y(t)] x_1^\varepsilon(t) \mathbb{E}(x_1^\varepsilon(t)) \right| dt \right\} \leq C\varepsilon\rho(\varepsilon), \quad (3.61)$$

and

$$\mathbb{E} \left\{ \int_s^T \left| Q^*(t) (\sigma_y(t))^2 (\mathbb{E}(x_1^\varepsilon(t)))^2 \right| dt \right\} \leq C\varepsilon\rho(\varepsilon), \quad (3.62)$$

$$\mathbb{E} \left\{ \int_s^T \left| \int_{\Theta} Q^*(t) (g_x(t, \theta))^2 (x_1^\varepsilon(t))^2 \mu(d\theta) \right| dt \right\} \leq C\varepsilon\rho(\varepsilon), \quad (3.63)$$

where  $\rho(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

**Proof.**

*Estimates of (3.60):* First we have

$$\begin{aligned} & \mathbb{E} \left\{ \int_s^T \left| \left[ \Psi^*(t) \delta f_x(t) + K^*(t) \delta \sigma_x(t) + \int_{\Theta} \gamma_t^*(\theta) \delta g_x(t, \theta) \mu(d\theta) \right] x_1^\varepsilon(t) \mathbf{I}_{\mathcal{E}_\varepsilon}(t) \right| dt \right\} \\ & \leq \mathbb{E} \left[ \int_s^T |\Psi^*(t) \delta f_x(t) x_1^\varepsilon(t) \mathbf{I}_{\mathcal{E}_\varepsilon}(t)| dt \right] + \mathbb{E} \left[ \int_s^T |K^*(t) \delta \sigma_x(t) x_1^\varepsilon(t) \mathbf{I}_{\mathcal{E}_\varepsilon}(t)| dt \right] \\ & \quad + \mathbb{E} \left[ \int_s^T \left| \int_{\Theta} \gamma_t^*(\theta) \delta g_x(t, \theta) \mu(d\theta) x_1^\varepsilon(t) \mathbf{I}_{\mathcal{E}_\varepsilon}(t) \right| dt \right] \\ & = \mathcal{I}_1^\varepsilon + \mathcal{I}_2^\varepsilon + \mathcal{I}_3^\varepsilon. \end{aligned} \quad (3.64)$$

Using (2.7) and estimates (3.13 with  $k = 1$ ), then from Cauchy-Schwarz inequality we get

$$\begin{aligned} \mathcal{I}_2^\varepsilon &= \mathbb{E} \left[ \int_s^T |K^*(t) \delta \sigma_x(t) x_1^\varepsilon(t) \mathbf{I}_{\mathcal{E}_\varepsilon}(t)| dt \right] \\ &\leq C \left[ \mathbb{E} \left( \sup_{t \in [s, T]} |x_1^\varepsilon(t)|^2 \right) \right]^{\frac{1}{2}} \left[ \mathbb{E} \left( \left( \int_s^T |K^*(t)| \mathbf{I}_{\mathcal{E}_\varepsilon}(t) dt \right)^2 \right) \right]^{\frac{1}{2}} \\ &\leq C\varepsilon^{\frac{1}{2}} \left[ \mathbb{E} \left( \left( \int_s^T |K^*(t)| \mathbf{I}_{\mathcal{E}_\varepsilon}(t) dt \right)^2 \right) \right]^{\frac{1}{2}} \\ &\leq C\varepsilon^{\frac{1}{2}} \left[ \mathbb{E} \left( \int_s^T |K^*(t)|^2 \mathbf{I}_{\mathcal{E}_\varepsilon}(t) dt \right) \right]^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} \leq C\varepsilon\rho_2(\varepsilon), \end{aligned} \quad (3.65)$$

where, also from (2.7) and *Dominated Convergence Theorem* we obtain

$$\rho_2(\varepsilon) = \left[ \mathbb{E} \left( \int_s^T |K^*(t)|^2 \mathbf{I}_{\mathcal{E}_\varepsilon}(t) dt \right) \right]^{\frac{1}{2}} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Similarly, we can prove estimate  $\mathcal{I}_1^\varepsilon$  then we get

$$\mathcal{I}_1^\varepsilon \leq C\varepsilon\rho_1(\varepsilon). \quad (3.66)$$

Let us turn to third term  $\mathcal{I}_3^\varepsilon$ . By using (2.7) and estimate (3.13 with  $k = 1$ ) with the help of Cauchy-Schwarz inequality we get

$$\begin{aligned} \mathcal{I}_3^\varepsilon &= \mathbb{E} \left[ \int_s^T \left| \int_{\Theta} \gamma_t^*(\theta) \delta g_x(t, \theta) \mu(d\theta) x_1^\varepsilon(t) \mathbf{I}_{\mathcal{E}_\varepsilon}(t) \right| dt \right] \\ &\leq C \left[ \mathbb{E} \left( \sup_{t \in [s, T]} |x_1^\varepsilon(t)|^2 \right) \right]^{\frac{1}{2}} \left[ \mathbb{E} \left( \left( \int_s^T \int_{\Theta} |\gamma_t^*(\theta)| \mathbf{I}_{\mathcal{E}_\varepsilon}(t) \mu(d\theta) dt \right)^2 \right) \right]^{\frac{1}{2}} \\ &\leq C\varepsilon^{\frac{1}{2}} \left[ \mathbb{E} \left( \int_s^T \int_{\Theta} |\gamma_t^*(\theta)|^2 \mathbf{I}_{\mathcal{E}_\varepsilon}(t) \mu(d\theta) dt \right) \right]^{\frac{1}{2}} \\ &\leq C\mu(\Theta)\varepsilon^{\frac{1}{2}} \left[ \mathbb{E} \left( \int_s^T \sup_{\theta \in \Theta} |\gamma_t^*(\theta)|^2 \mathbf{I}_{\mathcal{E}_\varepsilon}(t) dt \right) \right]^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} \leq C\varepsilon\rho_3(\varepsilon), \end{aligned} \quad (3.67)$$

Again, from (2.7) and *Dominated Convergence Theorem* we obtain

$$\rho_3(\varepsilon) = \left[ \mathbb{E} \left( \int_s^T \sup_{\theta \in \Theta} |\gamma_t^*(\theta)|^2 \mathbf{I}_{\mathcal{E}_\varepsilon}(t) dt \right) \right]^{\frac{1}{2}} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Finally, we set  $\rho(\varepsilon) = \rho_1(\varepsilon) + \rho_2(\varepsilon) + \rho_3(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  then the desired result follows immediately from combining (3.63)~(3.67). This completes the proof of (3.60).

*Estimates of (3.63):* First we have from assumption (H2) and by using (2.8) and estimate (3.13 with  $k = 1$ ) with the help of Cauchy-Schwarz inequality we get

$$\begin{aligned} &\mathbb{E} \left\{ \int_s^T \left| \int_{\Theta} Q^*(t) (g_x(t, \theta))^2 (x_1^\varepsilon(t))^2 \mu(d\theta) \right| dt \right\} \\ &\leq C\mu(\Theta) \mathbb{E} \left\{ \int_s^T \left| Q^*(t) \sup_{\theta \in \Theta} (g_x(t, \theta))^2 (x_1^\varepsilon(t))^2 \right| dt \right\} \\ &\leq C \left[ \mathbb{E} \left( \sup |x_1^\varepsilon(t)|^4 \right) \right]^{\frac{1}{2}} \left[ \mathbb{E} \left( \left( \int_s^T |Q^*(t)| dt \right)^2 \right) \right]^{\frac{1}{2}} \\ &\leq C\varepsilon\rho(\varepsilon), \end{aligned}$$

where  $\rho(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$

Using similar arguments developed above for estimates (3.61) and (3.62) which completes the proof of Lemma 3.5.

It worth mentioning that by combining the duality relations (3.6) and (3.7) in Lemma 3.1 together with Lemma 3.5 we get

$$\begin{aligned}
\mathbb{E}(\Psi(T)(x_1^\varepsilon(T) + x_2^\varepsilon(T))) &= \mathbb{E} \int_s^T (x_1^\varepsilon(t) + x_2^\varepsilon(t)) [(\ell_x(t) + \mathbb{E}(\ell_y(t)))] dt \\
&+ \mathbb{E} \int_s^T \left\{ \Psi(t) \delta f(t) + K(t) \delta \sigma(t) + \int_{\Theta} \gamma_t(\theta) \delta g(t, \theta) \mu(d\theta) \right\} \mathbf{I}_{\mathcal{E}_\varepsilon}(t) dt \\
&+ \mathbb{E} \int_s^T \left\{ \Psi(t) \mathcal{L}_t(f, x_1^\varepsilon) + K(t) \mathcal{L}_t(\sigma, x_1^\varepsilon) + \int_{\Theta} \gamma_t(\theta) \mathcal{L}_{t,\theta}(g, x_1^\varepsilon) \mu(d\theta) \right\} dt \\
&+ \tau(\varepsilon).
\end{aligned} \tag{3.68}$$

**Proof of Theorem 3.1.** By applying (3.2), (3.13) and Corollary 3.1 we get

$$\begin{aligned}
0 &\leq J^{s,\zeta}(u^\varepsilon(\cdot)) - J^{s,\zeta}(u^*(\cdot)) \\
&= \mathbb{E}[h(x^\varepsilon(T), \mathbb{E}(x^\varepsilon(T))) - h(x^*(T), \mathbb{E}(x^*(T)))] \\
&\quad + \mathbb{E} \int_s^T [\ell(t, x^\varepsilon(t), \mathbb{E}(x^\varepsilon(t)), u^\varepsilon(t)) - \ell(t, x^*(t), \mathbb{E}(x^*(t)), u^*(t))] dt \\
&= \mathbb{E}[h_x(x^*(T), \mathbb{E}(x^*(T)))(x_1^\varepsilon(T) + x_2^\varepsilon(T))] \\
&\quad + \mathbb{E}[h_y(x^*(T), \mathbb{E}(x^*(T)))(\mathbb{E}(x_1^\varepsilon(T)) + \mathbb{E}(x_2^\varepsilon(T)))] \\
&\quad + \mathbb{E} \int_s^T [\ell_x(t)(x_1^\varepsilon(t) + x_2^\varepsilon(t)) + \ell_y(t)(\mathbb{E}(x_1^\varepsilon(t)) + \mathbb{E}(x_2^\varepsilon(t)))] dt \\
&\quad + \mathbb{E} \int_s^T [\delta \ell(t) \mathbf{I}_{\mathcal{E}_\varepsilon}(t) + \mathcal{L}_t(\ell, x_1^\varepsilon)] dt + \mathbb{E}[\mathcal{L}_T(h, x_1^\varepsilon)] + \tau(\varepsilon),
\end{aligned}$$

then we get

$$\begin{aligned}
0 &\leq J^{s,\zeta}(u^\varepsilon(\cdot)) - J^{s,\zeta}(u^*(\cdot)) \\
&= \mathbb{E} \int_s^T [\delta \ell(t) \mathbf{I}_{\mathcal{E}_\varepsilon}(t) + \mathcal{L}_t(\ell, x_1^\varepsilon)] dt + \mathbb{E}[\mathcal{L}_T(h, x_1^\varepsilon)] \\
&\quad + \mathbb{E} \int_s^T [\ell_x(t) + \mathbb{E}(\ell_y(t))](x_1^\varepsilon(t) + x_2^\varepsilon(t)) dt \\
&\quad + \mathbb{E}\{[h_x(x^*(T), \mathbb{E}(x^*(T)) + \mathbb{E}(h_y(x^*(T), \mathbb{E}(x^*(T))))][x_1^\varepsilon(T) + x_2^\varepsilon(T)]\} + \tau(\varepsilon).
\end{aligned}$$

from (3.68) and the fact that  $\Psi^*(T) = -h_x(x^*(T), \mathbb{E}(x^*(T))) - \mathbb{E}(h_y(x^*(T), \mathbb{E}(x^*(T))))$  we obtain

$$\begin{aligned}
0 &\leq J^{s,\zeta}(u^\varepsilon(\cdot)) - J^{s,\zeta}(u^*(\cdot)) = \mathbb{E} \int_s^T [\delta \ell(t) \mathbf{I}_{\mathcal{E}_\varepsilon}(t) + \mathcal{L}_t(\ell, x_1^\varepsilon)] dt + \mathbb{E}[\mathcal{L}_T(h, x_1^\varepsilon)] \\
&\quad - \mathbb{E} \int_s^T \left\{ \Psi^*(t) \delta f(t) + K^*(t) \delta \sigma(t) + \int_{\Theta} \gamma_t^*(\theta) \delta g(t, \theta) \mu(d\theta) \right\} \mathbf{I}_{\mathcal{E}_\varepsilon}(t) dt \\
&\quad - \mathbb{E} \int_s^T \left\{ \Psi^*(t) \mathcal{L}_t(f, x_1^\varepsilon) + K^*(t) \mathcal{L}_t(\sigma, x_1^\varepsilon) \right. \\
&\quad \left. + \int_{\Theta} \gamma_t^*(\theta) \mathcal{L}_{t,\theta}(g, x_1^\varepsilon) \mu(d\theta) \right\} dt + \tau(\varepsilon).
\end{aligned}$$

Next by applying (2.9) we deduce

$$\begin{aligned}
0 &\leq J^{s,\zeta}(u^\varepsilon(\cdot)) - J^{s,\zeta}(u^*(\cdot)) \\
&= -\mathbb{E} \int_s^T \delta H(t) \mathbf{I}_{\mathcal{E}_\varepsilon}(t) dt \\
&\quad + \frac{1}{2} \mathbb{E} \left[ h_{xx}(x^*(T), \mathbb{E}(x^*(T))) (x_1^\varepsilon(T))^2 - \int_s^T H_{xx}(t) (x_1^\varepsilon(t))^2 dt \right] \\
&\quad + \tau(\varepsilon).
\end{aligned} \tag{3.69}$$

Now, from Lemma 3.4, then it easy to shows that

$$\begin{aligned}
&\frac{1}{2} \mathbb{E} [h_{xx}(x^*(T), \mathbb{E}(x^*(T))) x_1^\varepsilon(T)^2] \\
&= \mathbb{E} \int_s^T \left\{ \frac{1}{2} H_{xx}(t) (x_1^\varepsilon(t))^2 - \frac{1}{2} Q^*(t) \sigma_y^2(t) (\mathbb{E}(x_1^\varepsilon(t)))^2 \right. \\
&\quad - \frac{1}{2} Q^*(t) (\delta \sigma(t))^2 - \frac{1}{2} \int_{\Theta} Q^*(t) (\delta g(t, \theta))^2 \mu(d\theta) \mathbf{I}_{\mathcal{E}_\varepsilon}(t) \\
&\quad - \frac{1}{2} \int_{\Theta} \Gamma_t^*(\theta) (\delta g(t, \theta))^2 \mathbf{I}_{\mathcal{E}_\varepsilon}(t) \mu(d\theta) \\
&\quad - (\mathbb{E}(x_1^\varepsilon(t))) x_1^\varepsilon(t) [Q^*(t) f_y(t) + Q^*(t) \sigma_x(t) \sigma_y(t) + R^*(t) \sigma_y(t)] \\
&\quad \left. - \frac{1}{2} \int_{\Theta} Q^*(t) (g_x(t, \theta))^2 (x_1^\varepsilon(t))^2 \mu(d\theta) \right\} dt + \tau(\varepsilon),
\end{aligned} \tag{3.70}$$

using Lemma 3.5 together with (3.69) and (3.70) we obtain

$$\begin{aligned}
0 &\leq J^{s,\zeta}(u^\varepsilon(\cdot)) - J^{s,\zeta}(u^*(\cdot)) = -\mathbb{E} \int_s^T \delta H(t) \mathbf{I}_{\mathcal{E}_\varepsilon}(t) dt \\
&\quad - \frac{1}{2} \mathbb{E} \int_s^T Q^*(t) (\delta \sigma(t))^2 \mathbf{I}_{\mathcal{E}_\varepsilon}(t) dt \\
&\quad - \frac{1}{2} \mathbb{E} \int_s^T \int_{\Theta} Q^*(t) (\delta g(t, \theta))^2 \mathbf{I}_{\mathcal{E}_\varepsilon}(t) \mu(d\theta) dt \\
&\quad - \frac{1}{2} \mathbb{E} \int_s^T \int_{\Theta} \Gamma_t^*(\theta) (\delta g(t, \theta))^2 \mathbf{I}_{\mathcal{E}_\varepsilon}(t) \mu(d\theta) dt + \tau(\varepsilon),
\end{aligned}$$

then we get

$$\begin{aligned}
0 &\leq J^{s,\zeta}(u^\varepsilon(\cdot)) - J^{s,\zeta}(u^*(\cdot)) = -\mathbb{E} \int_s^T \delta H(t) dt \\
&\quad - \frac{1}{2} \mathbb{E} \int_s^T Q^*(t) (\delta \sigma(t))^2 \mathbf{I}_{\mathcal{E}_\varepsilon}(t) dt \\
&\quad - \frac{1}{2} \mathbb{E} \int_s^T \int_{\Theta} (Q^*(t) + \Gamma_t^*(\theta)) (\delta g(t, \theta))^2 \mathbf{I}_{\mathcal{E}_\varepsilon}(t) \mu(d\theta) dt + \tau(\varepsilon).
\end{aligned}$$

Finally by using (2.4) we deduce

$$\begin{aligned}
0 \leq & \mathbb{E} \int_s^T \left\{ -H(t, x^*, \mathbb{E}(x^*), u, \Psi^*(t), K^*(t), \gamma_t^*(\theta)) \right. \\
& + H(t, x^*, \mathbb{E}(x^*), u^*(t), \Psi^*(t), K^*(t), \gamma_t^*(\theta)) \\
& - \frac{1}{2} Q^*(t) (\sigma(t, x^*(t), \mathbb{E}(x^*(t)), u) - \sigma(t, x^*(t), \mathbb{E}(x^*(t)), u^*(t)))^2 \mathbf{I}_{\mathcal{E}_\varepsilon}(t) \\
& \left. - \frac{1}{2} \int_{\Theta} (Q^*(t) + \Gamma_t^*(\theta)) (g(t, x^*(t), u, \theta) - g(t, x^*(t), u^*(t), \theta))^2 \mathbf{I}_{\mathcal{E}_\varepsilon}(t) \mu(d\theta) \right\} dt \\
& + \tau(\varepsilon).
\end{aligned}$$

This completes the proof of Theorem 3.1.

**Conclusions.** In this paper, stochastic maximum principle for optimal stochastic control for systems governed by SDE of mean-field type with jump processes is proved. The control variable is allowed to enter both diffusion and jump coefficients and also the diffusion coefficients depend on the state of the solution process as well as of its expected value. Moreover, the cost functional is also of Mean-field type. When the coefficients  $f$  and  $\sigma$  of the underlying diffusion process and the cost functional do not explicitly depend on the expected value, *Theorem 3.1* reduces to stochastic maximum principle of optimality, proved in Tang et al., ([32], *Theorem 2.1*).

## Appendix

The following result gives special case of the Itô formula for jump diffusions.

**Lemma A1.** (*Integration by parts formula for jumps processes*) Suppose that the processes  $x_1(t)$  and  $x_2(t)$  are given by: for  $i = 1, 2$ ,  $t \in [s, T]$  :

$$\begin{cases} dx_i(t) = f(t, x_i(t), u(t)) dt + \sigma(t, x_i(t), u(t)) dW(t) \\ \quad + \int_{\Theta} g(t, x_i(t-), u(t), \theta) N(d\theta, dt), \\ x_i(s) = 0. \end{cases}$$

Then we get

$$\begin{aligned}
\mathbb{E}(x_1(T)x_2(T)) &= \mathbb{E} \left[ \int_s^T x_1(t) dx_2(t) + \int_s^T x_2(t) dx_1(t) \right] \\
&+ \mathbb{E} \int_s^T \sigma^*(t, x_1(t), u(t)) \sigma(t, x_2(t), u(t)) dt \\
&+ \mathbb{E} \int_s^T \int_{\Theta} g^*(t, x_1(t), u(t), \theta) g(t, x_2(t), u(t), \theta) \mu(d\theta) dt.
\end{aligned}$$

See Framstad et al., ([13], *Lemma 2.1*) for the detailed proof of the above Lemma.

**Proposition A1.** Let  $\mathcal{G}$  be the predictable  $\sigma$ -field on  $\Omega \times [s, T]$ , and  $f$  be a  $\mathcal{G} \times \mathcal{B}(\Theta)$ -measurable function such that

$$\mathbb{E} \int_s^T \int_{\Theta} |f(r, \theta)|^2 \mu(d\theta) dr < \infty,$$

then for all  $p \geq 2$  there exists a positive constant  $C = C(T, p, \mu(\Theta))$  such that

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_s^t \int_{\Theta} f(r, \theta) N(d\theta, dr) \right|^p \right] \leq C \mathbb{E} \left[ \int_s^T \int_{\Theta} |f(r, \theta)|^p \mu(d\theta) dr \right].$$

**Proof.** See Bouchard et al., ([3], *Appendix*).

**Lemma A2** (*Martingale representation theorem for jump processes*). Let  $\mathcal{G}$  be a finite-dimensional space and let  $m(t)$  be an  $\mathcal{G}$ -valued  $\mathcal{F}$ -adapted square-integrable Martingale. Then there exist  $q(\cdot) \in \mathbb{L}_{\mathcal{F}}^2([s, T], \mathcal{G})$  and  $g(\cdot, \cdot) \in \mathbb{M}_{\mathcal{F}}^2([s, T], \mathcal{G})$  such that

$$m(t) = m(s) + \int_s^t q(r) dW(r) + \int_s^t \int_{\Theta} g(r, \theta) N(d\theta, dr).$$

**Proof.** See Tang et al., ([32] *Lemma 2.3*).

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